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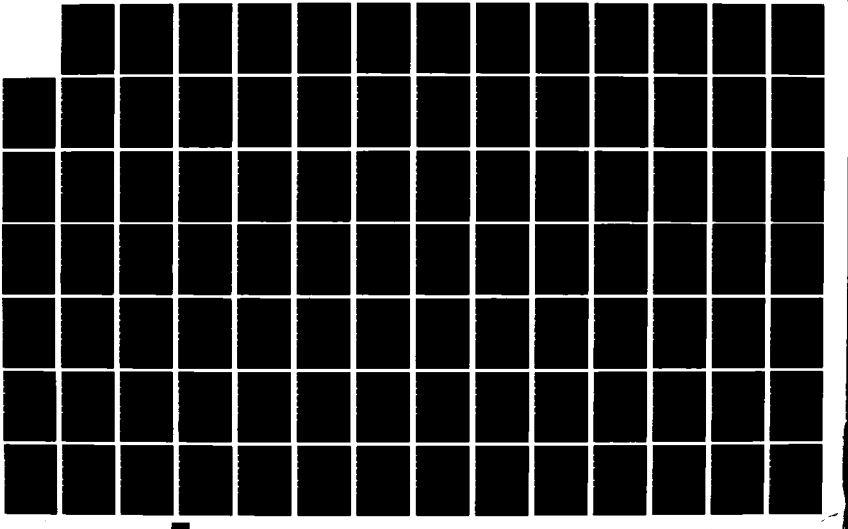

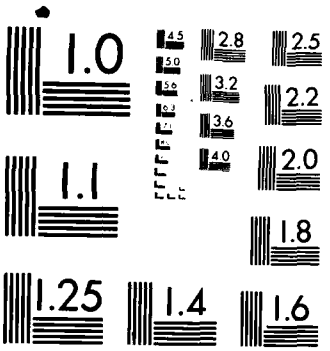

A FINITE ELEMENT ANALYSIS OF A CLASS OF PROBLEMS IN ELASTO-PLASTICITY WITH HIDDEN VARIABLES

J. Tinsley Oden

TICOM Report 85-11

Final Report

Army Research Office
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A general theory of finite elastoplasticity deformations is developed which makes use of the generalized Clarke-Rockafellar subdifferentials from non-convex optimization theory and the notion of internal state variables. The theory involves two fundamental potential functionals, the free energy and a general flow potential. An exact kinematical description of large elastic-plastic deformations is given, together with a complete thermodynamics. New finite element methods are derived and several example problems are solved to illustrate the generality and utility of the theory and the numerical schemes.

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FINAL REPORT

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Austin, Texas

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

0. INTRODUCTION

This document summarizes the results of a two-year research effort on a number of mathematical and numerical issues related to finite elastoplasticity. The Principal Investigator of the project was Professor J.T. Oden of The University of Texas.

The study set out to develop a general theory of finite elastoplasticity which makes use of standard continuum thermodynamic arguments, but which does not necessarily involve the assumption of existence of the yield function, which does not require conventional assumptions of convexity or differentiability of various functionals, which addresses and resolves the issue of proper decomposition of elastic and plastic deformation measures, and which under appropriate additional assumptions reduces to many theories known to be capable of describing infinitesimal deformations of elastoplastic solids. In addition, general forms of representation of elastoplastic constitutive equations were sought which would provide some measure of the effects of micromechanical changes in the constitution of the material. Finally, this general theory was used as the basis for the construction of new finite element methods for the calculation of large elastoplastic deformations, together with various algorithms and codes in order to produce numerical solutions of representative problems in this area.

Positive results on each one of the objectives were obtained. In particular, the following were accomplished:

1. Our theory is a theory of plasticity that does not necessarily involve the concept of stress: indeed, materials are fully characterized in a way consistent with thermodynamics by two stress potentials: the free energy functional and the generalized flow potential.

2. The theory can produce as special cases all of the classical theories of plasticity and hyper-elasticity. Yet it does not require the introduction of a yield function, nor does it necessarily involve assumptions of a normality rule of convexity.

3. The notion of the Clark/Rockafellar generalized sub-differential is used to provide a broad generalization of classical potential theory and make the potential arguments applicable to non-differential non-convex potentials. Isotropic function theory is used to further specialize acceptable forms of these functionals.

4. A correct kinematics of finite elastoplastic deformations has been produced, we believe for the first time. This involves the use of the polar decomposition theorem and a mathematically correct definition of the so-called stress-relaxed referenced state.

5. Since a correct kinematics is developed and potentials are used, the formulation make unnecessary the traditional difficulties of defining a correct, objective stress rate.

6. A new collection of numerical algorithms has been developed based on this theory. These have been developed in connection with finite element models of finite elastoplastic behavior. A working computer code has been developed and several example problems have been solved. While the code is primarily a plane strain code which employs bilinear quadrilateral elements, it has been used to solve a number of problems in finite deformation plasticity and elasticity and results are consistent with those obtained by other methods for special cases.

0.2 Publications

The following papers were published as a result of the reported work:

1. "Generalized Potentials in Finite Elastoplasticity," by S.J. Kim and J.T. Oden, International Journal of Engineering Science, Vol. 22, No. 11/12, 1984, pp. 1235-1257.

2. "Generalized Flow Potentials in Finite Elastoplasticity -- II. Examples," by S.J. Kim and J.T. Oden, International Journal of Engineering Science, Vol. 23, No. 5, 1985, pp. 515-530.

3. "Finite Element Analysis of a Class of Problems in Finite Elastoplasticity Based on the Thermodynamical Theory of Materials of Type-N," by S.J. Kim and J.T. Oden, Computer Methods in Applied Mechanics and Engineering, (in press).

4. "KABOD-A Finite Element Program for Large Elastoplastic Deformations Based on Generalized Flow Potentials," by S.J. Kim, TICOM Report 85-7, Austin, Texas, 1985.

5. "Theory of Finite Elastoplasticity," by S.J. Kim, TICOM Report 85-6, Austin, Texas, 1985.

0.3 Personnel

The following persons worked on this project:

1. Professor J. Tinsley Oden, Principal Investigator

2. Seung Jo Kim, Graduate Research Assistant, who received his Ph.D. degree in May, 1985.

3, Tsung-L. Lin, Philippe Devloo, L. Demkowicz, J. Bass, and Luis Faria provided a small portion of their time during various phases of the project.

0.4 Technical Discussion

A summary of most of the major results is given in the remainder of this report. These results include the development of kinematical relationships, the notion of generalized potentials and their thermodynamic consequences, representation results for flow potentials, finite element approximations, algorithms, and numerical results, some results on actual laboratory experiments and their correlation with the numerical solutions, and suggestions for future research.

CHAPTER I

INTRODUCTION

1.1 Introductory Comments

Classical plasticity theory is, in many ways, a product of inductive thought, growing as it did from attempts to model observed behavior of metals and soils under loading histories sufficient to create permanent deformaton. This approach toward the development of a mechanical theory is quite different from the deductive methods of modern continuum mechanics, where the framework for theories of material behavior is derived in a semi-axiomatic way from a small collection of universal postulates (laws of physics).

Over the last fifteen years, there have been numerous attempts to provide a thermodynamic basis for a plasticity theory sufficiently general to accommodate finite deformations but, at the same time, not inconsistent with either continuum thermodynamics or classical plasticity. While this volume of literature is too large to be adequately referenced here, we mention as significant examples, the works of GREEN and NAGHDI [1965], COLEMAN and GURTIN [1967], VALANIS [1971], ERINGEN [1962], LEE [1969, 1981], NEMAT-NASSER

[1979, 1982], HALPHEN and NGUYEN [1975], BHANDARI and ODEN [1973, 1975], RICE [1971], HAVNER and HILL [1982], and HAVNER [1982]; for more recent work and a more complete list of references in this general area, the proceedings edited by LEE and MALLET [1982], DESAI and GALLAGHER [1983], NEMAT-NASSER et. al [1984] and WILLAM [1984] can be consulted.

There are several fundamental features of classical plasticity that are particularly difficult to generalize in a way compatible with modern continuum mechanics: the basic kinematical descriptions such as the decompositions of strain and strain-rate measures into distinct elastic and plastic parts; an appropriate definition of stress rate; the existence of a yield function as distinct from the usual collection of constitutive equations required to characterize material behavior; so-called normality conditions; etc. In addition, one hopes that a theory which does capture and generalize these features will be capable of modeling aspects of the behavior of real materials which fall outside the realm of the classical theory, that it will be consistent with continuum thermodynamics, and that it will be of a form that lends itself to numerical approximation.

This report presents the development of a general theory of elastoplastic materials, which includes infinitesimal elastoplasticity as a special case, and which addresses and resolves each of

the difficulties mentioned above. In addition, particular special cases of the theory are considered which provide meaningful generalizations of classical elasticity and plasticity to cases of practical importance. In addition, a study of finite element approximations of the governing equations is given, together with new numerical algorithms and applications to representative problems.

1.2 Objectives and Scope

The first aim of this study is to develop a general theory of finite elastoplasticity which makes use of standard continuum thermodynamic arguments, does not necessarily involve the assumption of the existence of a yield function, does not require assumptions of convexity or differentiability of various functionals, addresses and resolves the issue of proper decomposition of various "elastic" and "plastic" deformation measures in a consistent way, and which, under appropriate additional assumptions, reduces to many theories known to be capable of describing infinitesimal deformations of elastoplastic solids.

Secondly, a general form of representation of the elastoplastic constitutive relations is sought which provides for some measure of the effects of micromechanical changes in the constitution of the material. This is provided for by regarding the principal

constitutive functional as dependent on internal state variables in addition to appropriate kinematical and thermal measures.

Thirdly, we wish to demonstrate that the theory lends itself to numerical approximation. Toward this goal, finite element approximations of the governing equations are derived along with a collection of new algorithms for treating large-deformation elastoplasticity problems. A code implementing these algorithms was developed for the Harris 800 supermini-computer, and used to test the algorithms on a number of representative example problems.

Finally, we wish to develop consistent finite element methods for the approximation of the equations and inequalities governing the general theory, together with new algorithms for the numerical analysis of representative problems.

We note that this list of objectives is a broad one, encompassing not only the creation of a new theory but also new numerical methods and actual applicatons to representative problems. In general, applications of such a general theory to concrete problems is impossible without some information on the specific form of constitutive equations and accompanying experimental data obtained from tests on actual materials. To provide this information for the problem studied analytically and numerically here, we construct slight generalizations of some flow potential functionals

proposed by BODNER and PARTOM [1972, 1975]. These data were then used in the determinaton of necessary material constants.

This report is divided into six chapters.

In Chapter 2, following this Introduction, mathematical and mechanical preliminaries are given. The principal mathematical machinery needed for the general techniques we employ is the theory of non-convex optimization and generalized subdifferentials advanced by CLARKE [1973, 1976, 1977] and ROCKAFELLAR [1979, 1980]. PANAGIOTOPOULOUS [1982, 1985] observed that these ideas had applicaton to certain plasticity problems. We outline the key mathematical concepts in Chapter 2 and provide a brief review of relevant ideas from non-convex analysis in an Appendix. Some essential micromechanical features of materials relevant to plasticity theories for metals are also reviewed with some interpretations of appropriate internal state variables.

In Chapter 3, the ideas and concepts introduced in Chapter 2 are used in the formal study of thermodynamic restrictions on the form of constitutive functions characerizing a general class of materials which we call "Materials of Type N", since they involve a generalized "normality" condition. A close examinaton of the kinematical variables and a proposal of new alternative kinematical measures of deformaton and deformaton rate are also presented.

In Chapter 4, general representations of free energy and flow potentials are developed. It is shown that classical theories of elastoplasticity can be recovered from our theory as special cases.

As an application of the theory, specific forms of the free energy and flow potential functionals are presented which are inspired by results of BODNER and PARTOM [1972, 1975]. To demonstrate the feasibility of our theory and show how to determine the material constants for our generalized Bodner-Partom material, actual physical tension tests were performed. A numerical scheme for using such sample experiments to determine material constants is also presented and discussed.

The finite element approximation of the governing equations is taken up in Chapter 5, where an incremental, total Lagrangian algorithm is described. Specific applications are also considered, including the numerical analysis of large-strain uniaxial stretching and compressing of a specimen and the crushing (upsetting) of rectangular billet in a metal forming simulation.

Among the new and special features of our results are the following:

Technically, our continuum theory does not require a concept of stress. The mechanical response of this class of materials (excluding heat flux) is completely characterized by

two potentials, the free energy and the flow potential, and only subgradients of these functionals appear in the equations of motion.

The potentials which play a predominant role in our formulation are true potentials only in a very generalized sense, made precise in Chapter 3; the potential functionals need not be convex nor need they be differentiable.

It is not necessary to assume the existence of a yield functon in order to describe yielding and elastoplastic deformation; however, if a yield function is known for a given class of materials, corresponding flow potentials can be constructed in a straightforward manner.

The various plasticity theories derivable from our general formulation are not necessarily "incremental" nor "rate-tpe" theories in the usual sense. This fact opens the door to several new families of numerical methods for solving finite deformation plasticity problems. We develop one such family of new schemes here and apply it to representative examples.

CHAPTER II

PRELIMINARIES

In this chapter, several mathematical notions prerequisite to our study and certain mechanical concepts on plastic deformation are presented.

2.1 Mathematical Preliminaries and Non-Convex Analysis

We begin with a summary of some recent results on non-convex analysis which follows the ideas of CLARKE [1973] and ROCKAFELLAR [1980]. Unless noted otherwise, V denotes a topological vector space, V^* the topological dual of V, and $\langle \cdot, \cdot \rangle$ a duality pairing on $V^* \times V$.

2.1.1 *Contingent and Tangent Cones*

Let K be a nonempty subset of a topological vector space V. Then the contingent cone to K at a point $u \in K$ is defined as the set

$$C_K(u) = \limsup_{\theta \to 0^+} \frac{1}{\theta}(K-u) \tag{2.1}$$

Likewise, the tangent cone to K at u is defined as the set

$$T_K(u) = \liminf_{\substack{u' \to u \\ u' \in K \\ \theta \to 0^+}} \frac{1}{\theta}[K-u'] \tag{2.2}$$

To interpret the notation used in (2.1) and (2.2), we use the concept of a limit superior (inferior) of a multifunction defined on a topological vector space given in the Appendix. Let Γ be a set-valued function from $[0,\infty) \times K$ into V such that

$$\Gamma(\theta,u) \equiv \frac{1}{\theta}(K - u) = \{v \in V \mid v = \frac{1}{\theta}(w-u), \quad w \in K,\ \theta \in [0,\infty)\} \tag{2.3}$$

Then

$$C_K(u) = \limsup_{\theta \to 0^+} \Gamma(\theta,u) = \bigcap_{\substack{A \in N(0) \\ \lambda>0}} \bigcup_{0 \in (0,\lambda)} [\Gamma(\theta,u) + A] \tag{2.4}$$

and

$$T_K(u) = \liminf_{\substack{u' \to u \\ \theta \to 0^+}} \Gamma(\theta,u') = \bigcap_{A \in N(0)} \bigcup_{\substack{B \in N(u) \\ \lambda > 0}} \bigcap_{\substack{u' \in K \cap B \\ \theta \in (0,\lambda)}} [\Gamma(\theta,u') + A] \tag{2.5}$$

where $N(0)$ and $N(u)$ denote collections of neighborhoods of 0 and u, respectively.

To visualize $C_K(u)$ and $T_K(u)$, we note that for $K \subset \mathbb{R}^N$,

$$C_K(\underset{\sim}{u}) = \{\underset{\sim}{v} \in \mathbb{R}^N \mid \exists\, \theta_k \to 0^+,\ \underset{\sim}{v}_k \to \underset{\sim}{v}, \text{ such that } u + \theta_k \underset{\sim}{v}_k \in K\} \tag{2.6}$$

$$T_K(\underset{\sim}{u}) = \{v \in \mathbb{R}^N \mid \forall \theta_k \to 0^+,\ \underset{\sim}{u}_k \to \underset{\sim}{u},$$

$$\underset{\sim}{u} \in K, \exists\ \underset{\sim}{v}_k \to \underset{\sim}{v} \text{ with } \underset{\sim}{u}_k + \theta_k \underset{\sim}{v}_k \in K\} \tag{2.7}$$

A two-dimensional case is illustrated in Fig. 2.1. Suppose $\underset{\sim}{u}$ terminates at a cusp in a non-convex set K, as shown. The entire plane can be represented as the union of four cones with vertex u: BOD, DOC, COA, and AOB, with O the terminix of $\underset{\sim}{u}$. Clearly, for any point $\underset{\sim}{v}$ inside the cone BOD $\cup$ DOC $\cup$ COA, it is always possible to find a sequence of positive numbers $\{\theta_k\}$ such that $\underset{\sim}{u} + \theta_k \underset{\sim}{v}_k \in K$ of any sequence $\underset{\sim}{v}_k \to \underset{\sim}{v}$. Outside of this cone (interior to AOB), it is impossible to find $\{\theta_k\}$ for which $\underset{\sim}{u} + \theta_k \underset{\sim}{v}_k \in K$. Hence,

$$C_K(\underset{\sim}{u}) = \text{BOD} \cup \text{DOC} \cup \text{AOB}$$

Similarly, pick a sequence $\underset{\sim}{u}_k \to \underset{\sim}{u}$ where $\underset{\sim}{u}_k$ is a sequence of vectors tracing out the arc EO on K. The legitimate vectors $\underset{\sim}{v}$ with sequences $\underset{\sim}{v}_k \to V$ such that $\underset{\sim}{u}_k + \theta_k \underset{\sim}{v}_k \in K$ as $\theta_k \to 0$ will be those in the half space BOD $\cup$ DOC. Similarly, for $\underset{\sim}{u}_k$ approaching $\underset{\sim}{u}$ along FO, we must choose $\underset{\sim}{v}$ in DOC $\cup$ COA. All other sequences $\underset{\sim}{u}_k \in K$, $\underset{\sim}{u}_k \to \underset{\sim}{u}$ yield acceptable $\underset{\sim}{v}$ in either of these half spaces. Thus, $T_K(\underset{\sim}{u})$ must represent the intersection:

$$T_K(\underset{\sim}{u}) = \text{DOC}$$

If K is convex, then

$$T_K(\underset{\sim}{u}) = C_K(\underset{\sim}{u})$$

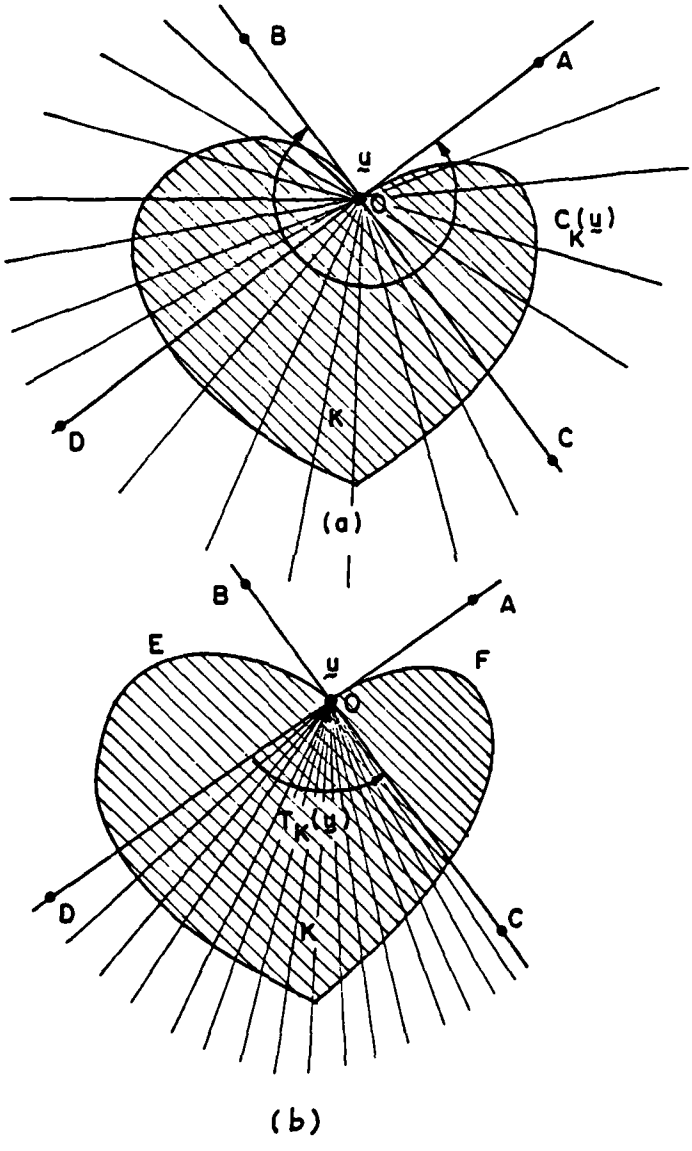


Figure 2.1 Interpretation of contingent cone $C_K(u)$ and tangent cone $T_K(u)$ for a non-convex set K.

2.1.2 *Normal Cone*

For $K \subset V$, $K \neq \emptyset$, the normal cone to K at u is a subset of the dual space V^* defined by

$$N_K(u) = \{u^* \in V^* | \langle u^*, v \rangle \leq 0 \quad \forall v \in T_K(u)\} \tag{2.8}$$

In two dimensions, $N_K(u)$ consists of the vectors through u which make obtuse angles with the vectors in $T_K(u)$, as shown in Fig. 2.2.

2.1.3 *Clark-Rockafellar Derivatives*

Let F be any extended real-valued function defined on V and let F be finite at a point $u \in V$. Then various types of subderivatives of F at u can be defined as follows.

• *Upper Subderivative.* The upper subderivative of F at u in direction v is defined as

$$D\uparrow F(u;v) = \underset{\substack{(u',\alpha)\downarrow(u,F(u)),v'\to v \\ \theta\to 0^+}}{\text{Lim sup inf}} \frac{1}{\theta}[F(u'+\theta v')-\alpha] \tag{2.9}$$

The notation Lim sup inf is defined in the Appendix, and by the notation

$$(u',\alpha) \downarrow (u,F(u))$$

we signify the convergence of a sequence $(u',\alpha) \in \text{epi } F$ to a point on the graph of F :

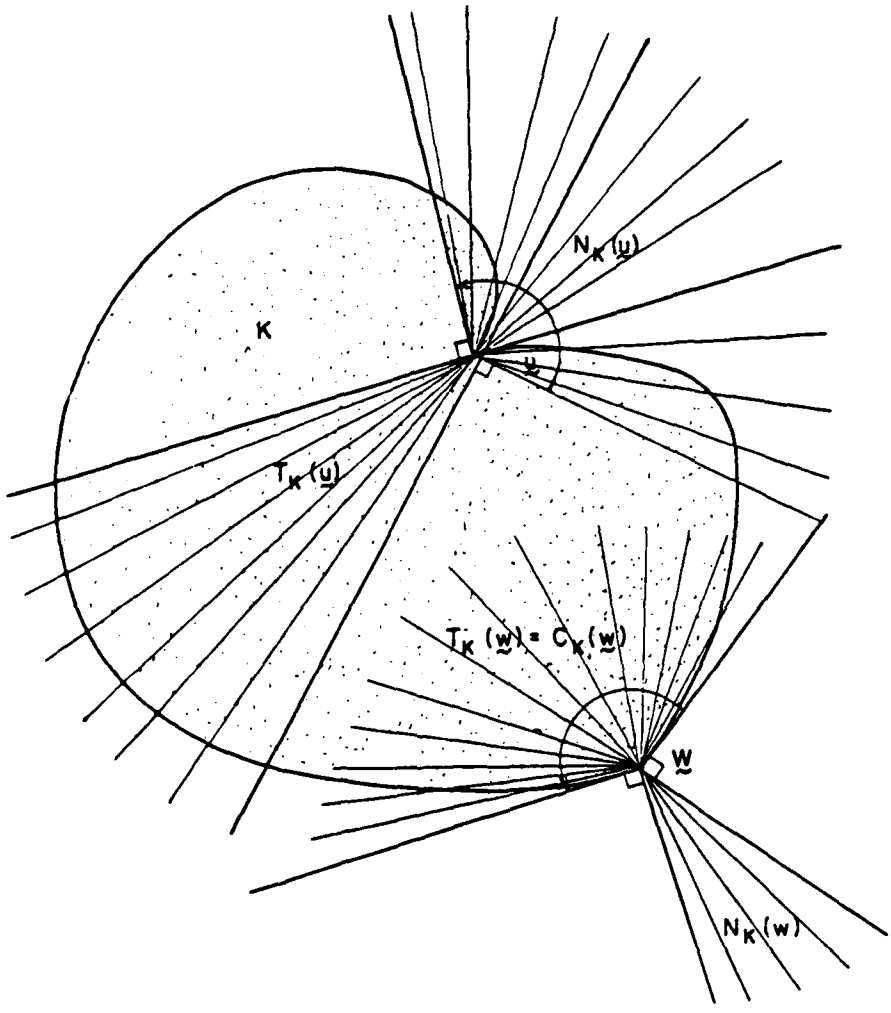


Figure 2.2 Normal and tangent cones at points of a set K.

$$(u',\alpha) \downarrow (u,F(u)) \Leftrightarrow (u',\alpha) \to (u,F(u)),$$

$$\alpha \geq F(u')$$

If F is l.s.c. (lower semicontinuous) at u, then (2.8) reduces to

$$D\uparrow F(u;v) = \operatorname*{Lim\,sup}_{\substack{(u',F(u')) \to (u,F(u)) \\ \theta \to 0^+}} \inf_{v' \to v} \frac{1}{\theta}[F(u'+\theta v')-F(u')] \tag{2.10}$$

• *Lower Subderivative.* The lower subderivative of F at u in direction v is defined by

$$D\downarrow F(u,v) = \operatorname*{Lim\,inf}_{\substack{(u',\alpha) \uparrow (u,F(u)), \\ \theta \to 0^+}} \sup_{v' \to v} \frac{1}{\theta}[F(u'+\theta v') - \alpha] \tag{2.11}$$

with

$$(u',\alpha) \uparrow (u,F(u)) \Leftrightarrow (u',\alpha) \to (u,F(u))$$

$$\alpha \leq F(u')$$

If F is u.s.c. at u,

$$D\downarrow F(u;v) = \operatorname*{Lim\,inf}_{\substack{(u',F(u')) \to (u,F(u)), \\ \theta \to 0^+}} \sup_{v' \to v} \frac{1}{\theta}[F(u'+\theta v')-F(u')] \tag{2.12}$$

The derivatives $D\uparrow F(u;v)$, $D\downarrow F(u;v)$ are referred to here as Clarke-Rockafellar derivatives (or C/R-derivatives for brevity).

To help understand the meaning of the C/R-derivatives, we consider the two examples shown in Figs. 2.3 and 2.4.

Example 1. Let f_1 and f_2 denote the real-valued functions,

$$f_1(x) = \begin{cases} x^2 - 4x + 4 & x \leq 1 \\ -x^2/2 + 2x - 1/2 , & x \geq 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} x^2 - 4x + 1 & x \leq 1 \\ -x^2/2 + 2x + 3/2 , & x \geq 1 \end{cases}$$

As we observe in Fig. 2.3, $f_1(x)$ is continuous at $x = 1$ but non-differentiable in classical sense and $f_2(x)$ is discontinuous but lower semicontinuous at $x = 1$. Since these functions are defined on $\mathbb{R}$, the definition of C/R-derivatives reduces to

$$D\uparrow f(x;y) = \limsup_{\substack{(x',f(x')) \\ \theta \to 0^+}} \frac{1}{\theta}[F(x' + \theta y) - F(x')] \quad (x,f(x))$$

and

$$D\uparrow f(x;y) = \liminf_{\substack{(x',f(x')) \to (x,f(x)) \\ \theta \to 0^+}} \frac{1}{\theta}[F(x' + \theta y) - F(x')]$$

Around $x = 1$, we have two subsequences of derivatives with $y=1$ which converges to 1 and -2 for both f_1 and f_2 and we easily conclude that

$$D\uparrow f_1(1;1) = 1 \qquad (= \text{"sup}\{1,-2\}\text{"})$$

$$D\uparrow f_2(1;1) = 1$$

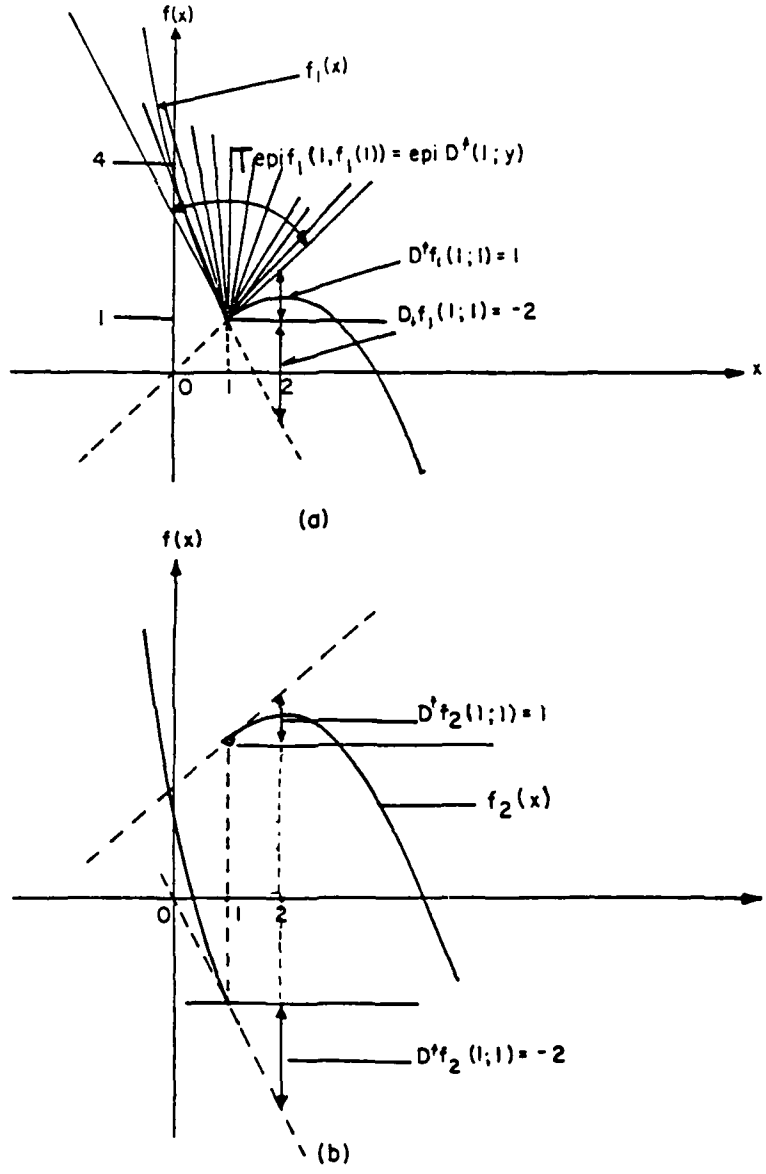


Figure 2.3 The upper (lower) subderivatives of (a) continuous but nondifferentiable and (b) discontinuous at x = 1.

$$D\uparrow f_1(1;1) = -2$$

$$D\uparrow f_2(1;1) = -2$$

We remark that the convergence $(x',\alpha) \downarrow (x,f(x))$ in the definition of C/R-derivatives (2.8) and the definition of tangent cone (2.2) lead to the fact that the epigraph of the function $y \to D\uparrow f(x;y)$ is the tangent cone $T_{epi\ f}(x,f(x))$ which is shown in Fig. 2.3a

Example 2. A better example of C/R-derivatives can be constructed in $\mathbb{R}^2$. Consider the lower semicontinuous function $f = f(x,y)$ shown in Fig. 2.4. The numbers indicated in the figure are intended to mean the following:

- The slope of line GH at G in the direction $\underset{\sim}{v} = (1,0)$ is +0.5
- The slope of line EF at F in the direction $\underset{\sim}{v} = (1,0)$ is -0.5
- The slope of line CD at C in the direction $\underset{\sim}{v} = (1,0)$ is -0.8
- The slope of line AB at B in the direction $\underset{\sim}{v} = (1.0)$ is +0.2

Let us calculate C/R-derivative at the origin $\underset{\sim}{u} = (0,0)$ with direction $\underset{\sim}{v} = (1,0)$. Recall that

$$D\uparrow f((0,0);(1,0)) = \limsup_{\substack{(\underset{\sim}{u}',F(\underset{\sim}{u}')) \to ((0,0),f(0,0)) \\ \theta \to 0^+}} \inf_{\underset{\sim}{v}' \to \underset{\sim}{v}} \left\{\frac{1}{\theta}(f(\underset{\sim}{u}' + \underset{\sim}{v}') - f(\underset{\sim}{u}'))\right\}$$

Along the direction of $\underset{\sim}{v} = (1,0)$, we choose two sequences approaching $\underset{\sim}{u} = (0,0)$ from either positive or negative x-axis:

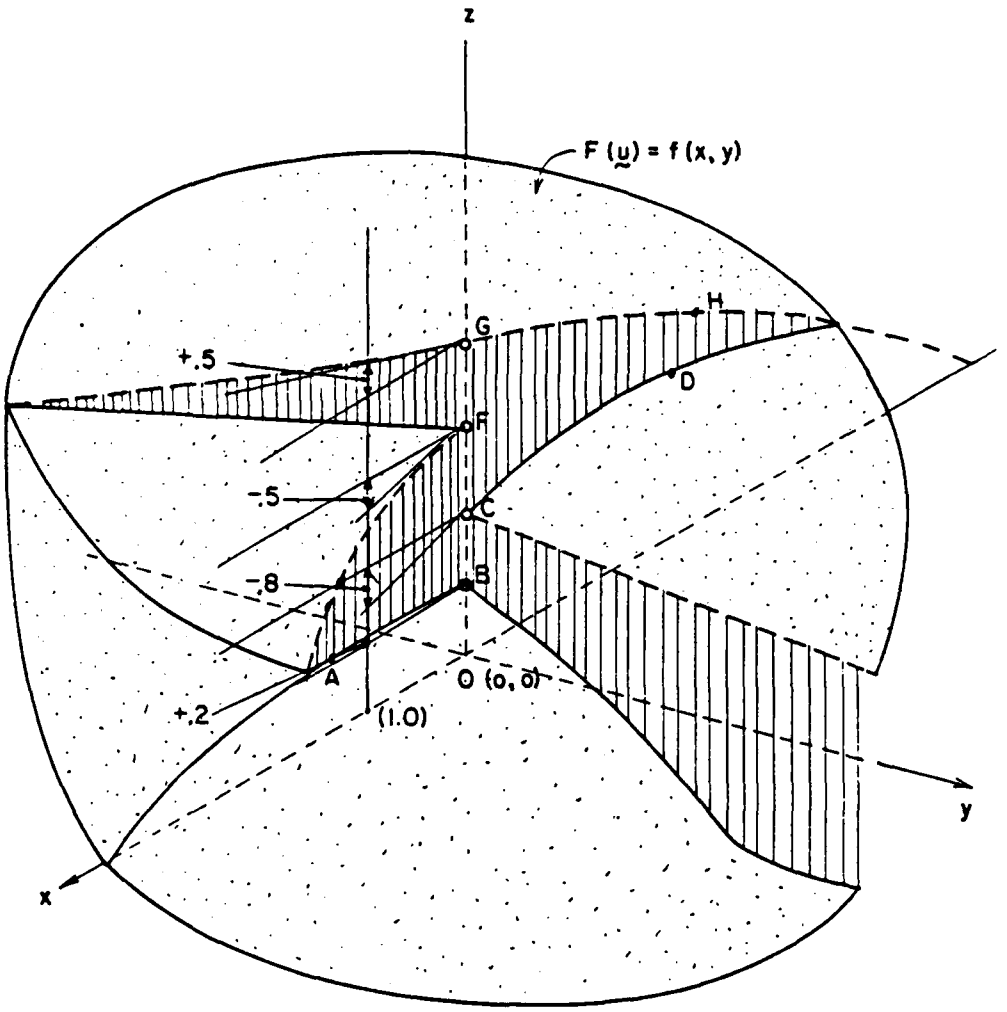


Figure 2.4 The upper (lower) subderivative of the lower semi-continuous function $F(\underset{\sim}{u})$.

$\underset{\underset{\sim}{v}' \to \underset{\sim}{v}}{\text{Lim inf}}\ Df(\underset{\sim}{u}',\underset{\sim}{v}')$ taken from the positive side of x-axis will be a sequence of slopes along EF and $\underset{\underset{\sim}{v}' \to \underset{\sim}{v}}{\text{Lim inf}}\ Df(\underset{\sim}{u}',\underset{\sim}{v}')$ taken from the negative side of x-axis will be the slopes along DC, where $Df(u',v') = \frac{1}{\theta}[f(\underset{\sim}{u}' + \theta\underset{\sim}{v}') - f(\underset{\sim}{u}')]$. Next we take $\underset{(u',f(u')) \downarrow ((0,0),f(0,0))}{\text{lim sup}}$ of the sequences of slopes and then finally the slope at F along the curve EF as $D\uparrow f((0,0);(1,0))$. Similarly, we can get

$$D\downarrow f((0,0);(1,0)) = \text{the slope at B along AB.}$$

Quantitatively

$$D\uparrow f((0,0);(1,0)) = -.5$$

and

$$D\downarrow f((0,0);(1,0)) = +.2$$

2.1.4 *Generalized Subdifferentials*

The subdifferential $\partial F(u)$ of a function F at u is a well known concept in convex analysis (see e.g., EKERLAND and TEMAM [1976] or ODEN [1985]). By using the subderivatives defined in (2.8) and (2.10), we define the generalized subdifferential of $F:V \to \mathbb{R}$, at a point u where F(u) is finite, as the set

$$\bar{\partial}F(u) = \{u^* \in V^* \mid \langle u^*,v\rangle \leq D\uparrow F(u;v) \quad \forall\, v \in V\} \tag{2.13}$$

Now we list two useful theorems due to ROCKAFELLAR [1980].

Theorem 2.1. Let F be any extended real-valued function on V, and let u be any point at which F is finite. Then $\bar{\partial}F$ is a weak*-closed convex subset of V* and

$$\overline{\partial}F(u) = \{u^* \in V^* \mid (u^*,-1) \in N_{epiF}(u,F(u))\} \tag{2.14}$$

If $D\uparrow F(u;0) = -\infty$, then $\overline{\partial}F(u)$ is empty, but otherwise $\overline{\partial}F(u)$ is nonempty and

$$D\uparrow F(u;v) = \sup \{\langle u^*,v\rangle \mid u^* \in \overline{\partial}F(u)\} \text{for all } v \in V \tag{2.15}$$

Theorem 2.2. If F is a convex function on V, then $\overline{\partial}F(u)$ agrees with the subgradient set in the sense of convex analysis:

$$\begin{aligned}\overline{\partial}F(u) &= \partial F(u) \\ &= \{u^* \in V^* \mid \langle u^*,v\rangle \leq F'(u;v), \forall v \in V\} \\ &= \{u^* \in V^* \mid \langle v-u,v^*\rangle \leq F(u)-F(u)\ \forall\ v \in V\}\end{aligned} \tag{2.16}$$

Here $F'(u;v) = \lim_{t\to 0^+} (F(u+tv) - F(u))/t$ is called the one-sided directional derivative which exists for all v when F is convex (although it may be infinite).

Remark. If F(u) is a characteristic function with respect to a set K, i.e., if

$$F(u) = \psi_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{if } u \notin K \end{cases}$$

then

$$\begin{aligned}\overline{\partial}F(u) &= \{u^* \in V^* \mid \langle u^*,v\rangle \leq D\uparrow\psi_K(u;v)\ \forall\ v \in V\} \\ &= N_K(u)\end{aligned} \tag{2.17}$$

This fact can be more easily visualized in the case of convex F, i.e.,

$$\partial\psi_K(u) = \{u^* \in V^* \mid \langle u^*, v-u\rangle \leq \psi_K(v) - \psi_K(u), \ \forall\, v \in K\}$$

$$= \{u^* \in V^* \mid \langle u^*, v-u\rangle \leq 0, \forall\, v \in K\}$$

$$= N_K(u)$$

since $\psi_K(v) = \psi_K(u) = 0$.

2.2 Mechanical Preliminaries

2.2.1 *Classical Continuum Plasticity*

Classical plasticity theory rests on the assumption of the existence of a convex function $F : \mathbb{M} = \mathbb{R}^3 \times \mathbb{R}^3 \to [0,-\infty)$ of the stress tensor $\underset{\sim}{\sigma}$, called the yield function of the material, which has the property that plastic flow at a particle X of the material is signaled whenever $F(\underset{\sim}{\sigma}(X)) = 0$ and $\frac{\partial F}{\partial \underset{\sim}{\sigma}} : \dot{\underset{\sim}{\sigma}} \geq 0$; otherwise the deformation at X is elastic (where $\underset{\sim}{A}:\underset{\sim}{B} = \text{tr}\, \underset{\sim}{A}^T\underset{\sim}{B}$):

$$\left.\begin{array}{l} F(\underset{\sim}{\sigma}(\underset{\sim}{X})) < 0 \\ F(\underset{\sim}{\sigma}(\underset{\sim}{X})) = 0 \text{ and } \frac{\partial F}{\partial \underset{\sim}{\sigma}} : \overset{\circ}{\underset{\sim}{\sigma}} < 0 \end{array}\right\} \text{ elastic deformation}$$

$$F(\underset{\sim}{\sigma}(\underset{\sim}{X})) = 0 \text{ and } \frac{\partial F}{\partial \underset{\sim}{\sigma}} : \overset{\circ}{\underset{\sim}{\sigma}} \geq 0 \Rightarrow \text{plastic flow} \qquad (2.18)$$

The only stress states admissible in such theories are those for which $F(\underset{\sim}{\sigma}) \leq 0$ or, equivalently, those stresses which belong to the convex set

$$K = \{\underset{\sim}{\sigma} \in \mathbb{M} \mid F(\underset{\sim}{\sigma}) \leq 0\} \qquad (2.19)$$

In general, the yield function may contain several parameters characterizing the state of material, but during plastic deformation

these parameters are understood to change in such a way that F remains equal to zero while plastic flow continues. These material parameters will also be expected to vary with temperature or other thermodynamic variables. The general yield condition relates only to the state of stress at a material particle, and, irrespective of whether the mechanical response is elastic or plastic, does not depend on the stress gradients.

It is further assumed in the classical theory of plasticity that yield is unaffected by hydrostatic stress, a situation generally in accord with experiment for moderate stress levels. With this assumption it is possible to represent the yield function for an isotropic material as a function of the invariants of the stress deviator in the form

$$\hat{F}(\underset{\sim}{\sigma}) = \bar{F}\ (J_2, J_3) \tag{2.20}$$

Where the invariants

$$J_2 = 1/2\ \mathrm{tr}\ (\underset{\sim}{\sigma}')^2,\ J_3 = \det \underset{\sim}{\sigma}' \text{ with } \underset{\sim}{\sigma}' = \underset{\sim}{\sigma} - 1/3\ \mathrm{tr}\ \underset{\sim}{\sigma} \tag{2.21}$$

(More detailed discussions of representations of response functions are given in Chapter IV.)

The infinitesimal strain tensor ε is representable as the sum of an elastic part $\underset{\sim}{\varepsilon}^e$ and a plastic strain $\underset{\sim}{\varepsilon}^P$, and its time rate-of-change is

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^e + \dot{\underset{\sim}{\varepsilon}}^P \tag{2.22}$$

It is meaningful to assume the existence of a plastic flow potential $\psi : \mathbb{M} \to R$ which is convex and l.s.c. and which has the property that

$$\dot{\underset{\sim}{\varepsilon}}^{p} \in \partial \psi(\underset{\sim}{\sigma}) \tag{2.23}$$

In particular, the indicator function ψ_K of the set K may define a specific flow potential as follows:

$$\psi_K(\underset{\sim}{\sigma}) = \begin{cases} 0 & \text{if } \underset{\sim}{\sigma} \in K \\ +\infty & \text{if } \underset{\sim}{\sigma} \notin K \end{cases} \tag{2.24}$$

Note that ψ_K is l.s.c. on $\mathbb{M}$ and that ψ_K is convex if F is convex.

From (2.23) and the definition of the subdifferential

$$\dot{\underset{\sim}{\varepsilon}}^{P} \in \partial \psi_K (\bar{\underset{\sim}{\sigma}}) \tag{2.25}$$

for some particular stress $\bar{\underset{\sim}{\sigma}}$, and this implies that

$$\left\langle \dot{\underset{\sim}{\varepsilon}}^{P}, \underset{\sim}{\sigma} - \bar{\underset{\sim}{\sigma}} \right\rangle \geq 0 \quad \forall \underset{\sim}{\sigma} \in K \tag{2.26}$$

This result, of course, is the classical normality condition which extablishes that the strain rate is normal to the yield surface or lies in the normal cone of the yield surface at corners. (See Fig. 2.5).

2.2.2 *Micromechanical Plasticity*

It is now a well-known fact that the permanent deformation of crystalline material is caused by microscopic defects in the crystalline structures. A brief review of some features of these underlying physical phenomena is useful at this point. For a

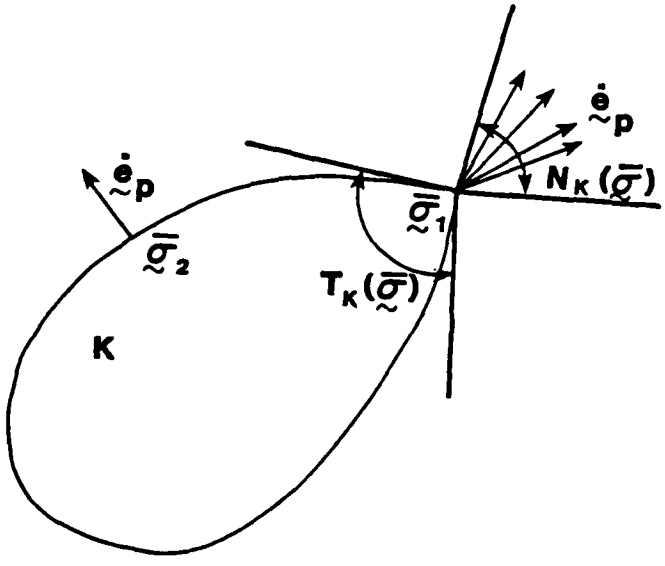


Figure 2.5 A convex yield surface and normality condition

detailed treatment, COTTRELL [1958], WEERTMAN and WEERTMAN [1964], and LARDNER [1974] can be consulted.

When most materials solidify from a melt, the constituent atoms arrange themselves in a highly regular periodic array called a crystal. They do this simply because the total energy, kinetic and potential, of the individual atoms, is less when the atoms are arranged in a coordinated manner than when they are randomly distributed. This energy minimum changes for different atomic constituents, so that the stable crystal structure formed during solidification vary for each material. The exact nature of the structure is determined by the strength and directionality of the interatomic forces.

A high degree of geometrical perfection in crystals is rarely found in either natural or synthetic crystals. In all real materials, crystals contain a small number of atomic irregularities or imperfections which do little to change the normal atomic arrangement but play an important role in modifying its physical properties. The defects of most importance in determining physical behavior are dislocations, point defects, and the structure of grain and phase boundaries.

The dislocations act as the carriers of plastic deformation through the motion of slip, and produce most of the noticeable changes of the internal mechanical state during the deformation. Point

defects, such as vacancies and solute atoms of a different species usually introduce a strong viscous effect into the deformation behavior. Finally, the structure of the grain and phase boundaries not only can increase the resistance against plastic flow but they can also be additional sources of viscous behavior.

For a perfect single crystalline material, one would expect elastic-perfectly-plastic response because when the shearing stress reaches some critical value it causes a continuous slipping motion as there are no obstacles which may interrupt or stop the shearing motion. But, as mentioned earlier, most materials do have barriers like grain boundaries, Frank networks, stationary substructures, etc. To break these barriers, the critical stress (or, phenomenologically, the yield stress) must be attained. This increased stress produces more barriers and again, an increased stress is required to break dislocation barriers, and so on. This phenomenon is called work-hardening.

Generally, in single crystalline models, we have four distinct stages in the stress-strain curve,as seen in Fig. 2.6. After an elastic stage, one often observes an "easy-glide" stage (Stage I in the figure) during which all the free (mobile) dislocations move and a large amount of plastic strain is realized. Stage II is called the work-hardening stage, and unit dislocations are generated from, say, Frank-Read sources and interactions between dislocations making barriers like Lomer-Cottrell locks. In Stage III, breakdowns of

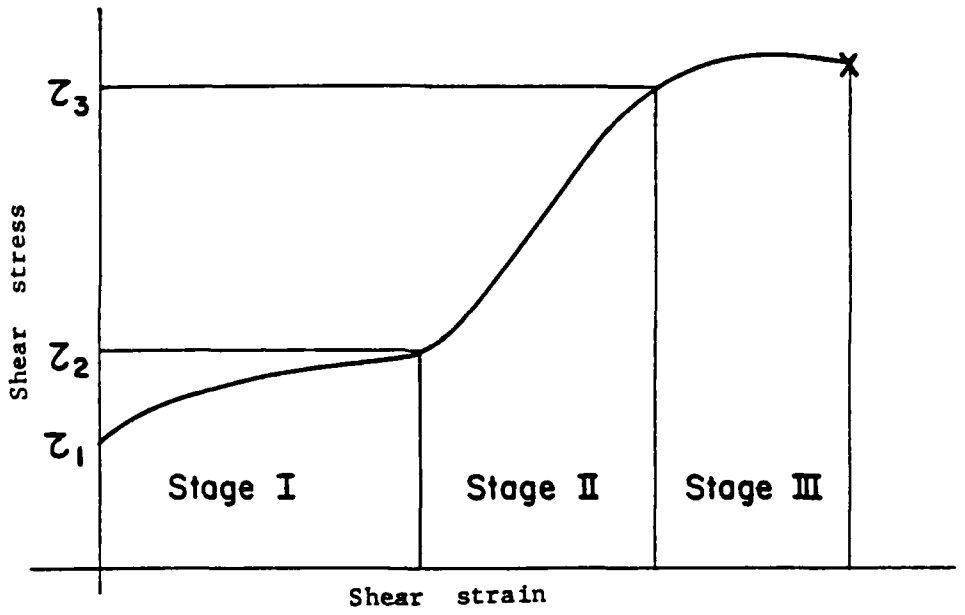


Figure 2.6 Stress-strain curve of single crystal material

these barriers may occur, giving very low hardening with some plastic strain. For more detailed descriptions of these concepts, see DIETER [1976] or WILKOV [1983].

In polycrystalline materials, the easy glide stage is not observed as frequently as in the single crystal since the movements of dislocations are stopped more readily by a large density of built-in obstacles due to the nature of the grain boundaries. Therefore the usual stress-strain curve (Fig. 2.7), i.e., a linear elastic region and a strain hardening region, is obtained. In fact, the easy glide stage is included in the elastic region since the flow of dislocation and plastic strains are ignorable.

2.2.3 *Internal State Variables*

From the previous section, we recognize that changes of internal structures in materials causes permanent deformations. A key question is how can one utilize this micromechanical information in formulating a continuum theory of elasto-plasticity. One approach, which has provided for some progress toward a general theory, involves the use of continuum theories which employ "hidden variables" or "internal state variables" as a measure of physical changes in the microstructure. Some of these changes can be those ordinarily associated with plastic deformation. These so-called internal state variables can provide measures of slip rearrangements of crystallographic planes through dislocation motions, inelastic behavior

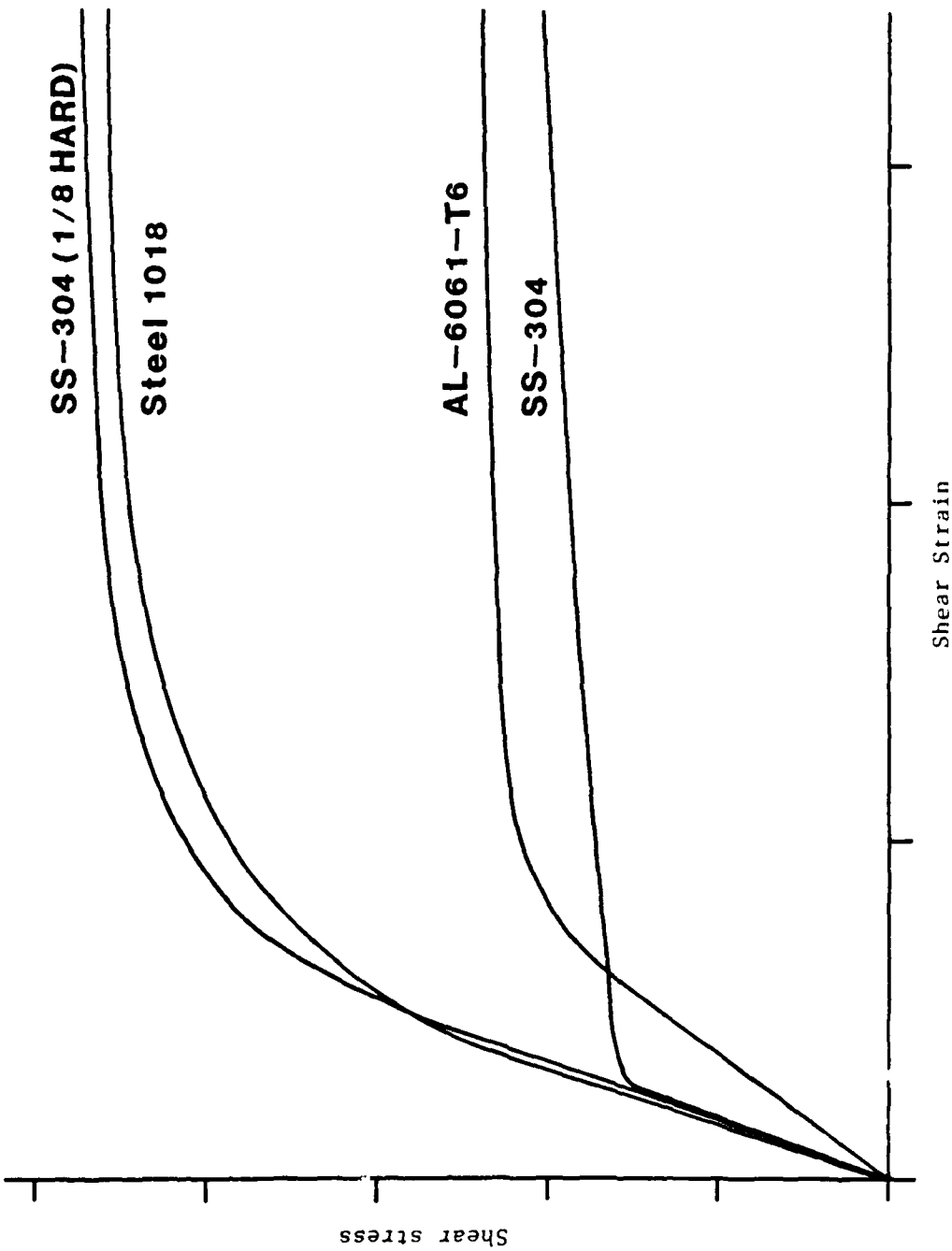


Figure 2.7 Stress-strain curves of polycrystalline materials

arising from twinning in crystals, grain boundary sliding, and stress-induced phase transformations.

The idea of developing a continuum theory of thermodynamics using internal state variables was actually proposed by MEIXNER [1953] and independently by BIOT [1954]. A general thermomechanical theory of materials with internal state variables was developed by COLEMAN and GURTIN [1967] and used in attempts by PERZYNA [1963], PERZYNA and WOJNO [1968], ODEN and BHANDARI [1973], KRONER [1962], KRONER and TEODOSIU [1974], RICE [1971, 1975] and others to develop a plasticity theory. Among the advantages of such an approach are that it simplifies the modeling of history effects in elastic-plastic materials since they are, in essence, accounted for by an evolution equation for the internal state variables. There are three categories of classifications of internal state variables.

The first employs the classical theory of elastoplasticity and its variations, e.g., MROZ [1973, 1981] and NECAS and HLAVACEK [1981]. In this approach, the hardness variables in the flow theory and the back stress in the kinematic hardening are the internal state variables.

In the second approach, a direct application of micromechanical structure is modeled, e.g., KRONER [1962], KRONER and TEODOSIU [1974], RICE [1971], ACHENBACH, MULLER and WILMANSKI [1981] and WENG [1980, 1981] Various measures of dislocation density, degree of impurity, and twinning modes can be identified as internal state variables.

The third approach employs a combination of the ideas of internal state variables and the phenomenological behavior as derived from experiments, e.g., HART [1976, 1979], BODNER and PARTOM [1972, 1975], etc. This approach combines features of the other two. More complex constitutive equations with more variables which are drawn from the phenomenological results are involved. A simple version in this category may be an elastoplasticity theory (ARAVAS [1985]) with micro-void implementation in which a classical yield function (or flow potential), e.g., von Mises, is modified by introducing the volume fraction of micro-voids in the material. This particular theory, which may find rare applications for engineering materials undergoing large deformations (see FUNG, BURNS and LIND [1973]), results in a pressure dependent plasticity theory which may be capable of modeling a class of porous materials. Later we utilize this third approach in an application of the theory of Materials of Type N.

CHAPTER III

THE MATERIALS OF TYPE N

In this chapter, a general continuum theory is developed for a hypothetical class of materials which undergo large plastic deformations.

3.1 Kinematical Considerations in Finite Deformations

While somewhat difficult to define with flawless mathematical precision, the basic idea of a finite "plastic" deformation of a material is heuristically clear: it is the "irrecoverable" part of the deformation of a material subjected to a loading cycle. One imagines that a material in a reference stress state $\underset{\sim}{\sigma}_0$ at a particle X is subjected to motions which carry the stress at this particle through a history which eventually returns the stress to the original state $\underset{\sim}{\sigma}_0$. If, at the conclusion of this stress cycle, the local state of deformation at X, however one chooses to measure it, differs from what it was before the stress cycle, then a portion of the deformation was not "recovered" and this is dubbed the "plastic" deformation. It is clear that these ideas are local in character; they may have meaning only in a local neighborhood of a material particle or for bodies in states of homogeneous deformation. Having these ideas in our mind, we offer in this section alternative decompositions of

the deformaton gradient to the multiplicative and the additive ones proposed by LEE [1969, 1981] and NEMAT-NASSER [1979, 1982], respectively.

Let us consider the motion of material body B relative to a fixed configuration $C_0 \subset \mathbb{R}^N$ ($N \leq 3$), which is defined by the map κ_0; $B \to \mathbb{R}^N$, $\underset{\sim}{X} = \kappa_0(X)$, where X is a material particle. The spatial position $\underset{\sim}{x}$ of a particle X at time t is then given by a relation of the type

$$\underset{\sim}{x} = \underset{\sim}{\chi}(X, t) \tag{3.1}$$

with $\underset{\sim}{X} \in \kappa_0(B)$, $t \geq 0$, and $\underset{\sim}{\chi}$ a continuous invertible map from C_0 into $\mathbb{R}^N$. The deformation gradient tensor $\underset{\sim}{F}$ at $\underset{\sim}{X}$ at time t is defined by

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{\chi}}{\partial \underset{\sim}{X}} \tag{3.2}$$

Let $N(\underset{\sim}{X})$ denote a small material neighborhood of particle X. The motion of the body carries $N(\underset{\sim}{X})$ from the reference configuration C_0 to the current configuration C_t. Let the Cauchy stress $\underset{\sim}{\sigma}$ at any particle $\underset{\sim}{A} \in N(\underset{\sim}{X})$ in C_0 be denoted $\underset{\sim}{\sigma}(\underset{\sim}{A}, 0)$ with

$$\underset{\sim}{\sigma}(\underset{\sim}{A}, 0) = \underset{\sim}{\sigma}_0(\underset{\sim}{X}) + \underset{\sim}{\omega}(\Delta \underset{\sim}{X}) \qquad \forall\ \underset{\sim}{A} \in N(\underset{\sim}{X})$$

where $\Delta \underset{\sim}{X} = \underset{\sim}{A} - \underset{\sim}{X}$. If $\underset{\sim}{\sigma}$ is continuous at $\underset{\sim}{X}$, then

$$\lim_{\|\Delta \underset{\sim}{X}\| \to 0} \frac{\|\underset{\sim}{\omega}(\Delta \underset{\sim}{X})\|}{\|\Delta \underset{\sim}{X}\|} = 0$$

$\|\cdot\|$ denotes the Euclidean norms. We shall refer to $\underset{\sim}{\sigma}_0(X)$ as the initial stress at particle $\underset{\sim}{X}$. For simplicity, we omit other variables (such as temperature, etc.) that could also be listed in defining the "initial state" of the material.

During an interval of time $[0,t]$, $t > 0$, the stress at particles in $N(\underset{\sim}{X})$ are part of the stress history

$$\underset{\sim}{H}_t(\underset{\sim}{A}) = \{\underset{\sim}{\sigma}(\underset{\sim}{A}, \tau) \mid \underset{\sim}{A} \in N(\underset{\sim}{X}), \quad 0 \leq \tau \leq t\}$$

and the configurations of $N(\underset{\sim}{X})$ are denoted $\underset{\sim}{\chi}(N(\underset{\sim}{X}), \tau)$, $0 \leq \tau \leq t$.

In addition to the actual stress history $\underset{\sim}{H}_t$, we consider any stress history $\underset{\sim}{H}^R_t$, corresponding to a relaxation of the stress at $\underset{\sim}{X}$, such that

$$\underset{\sim}{H}^R_t(\underset{\sim}{A}) \subset \underset{\sim}{H}_t(\underset{\sim}{A}), \quad \underset{\sim}{H}^R_t \in \underset{\sim}{H}_0$$

where H_0 is the family of all stress histories terminating at $\underset{\sim}{\sigma}_0$:

$$\underset{\sim}{H}^R_t(\underset{\sim}{A}) \in \underset{\sim}{H}_0 \Rightarrow \underset{\sim}{\sigma}(\underset{\sim}{A}, t) = \underset{\sim}{\sigma}_0(\underset{\sim}{A}, 0)$$

For $\underset{\sim}{p}$ the values of a continuous map of $N(\underset{\sim}{X})$ into $\mathbb{R}^N$, we denote by C_p any configuration of B for which the stress history at $\underset{\sim}{A} \in N(\underset{\sim}{X})$ is $\underset{\sim}{H}^R_t(\underset{\sim}{A})$. Thus, the introduction of a (possibly unattained) configuration C_p provides for the familiar device of comparing the geometries of the body in C_p with C_0 to define plastic deformation. A typical illustration of this is given in Fig. 3.1, wherein $N(\underset{\sim}{X})$ is shown in C_0, C_t, and C_p together with differential vectors $d\underset{\sim}{X}$, $d\underset{\sim}{x}$, and $d\underset{\sim}{p}$ [cf. LEE [1981, p. 862]].

We shall now proceed to construct decompositions of the total deformation into elastic and plastic parts.

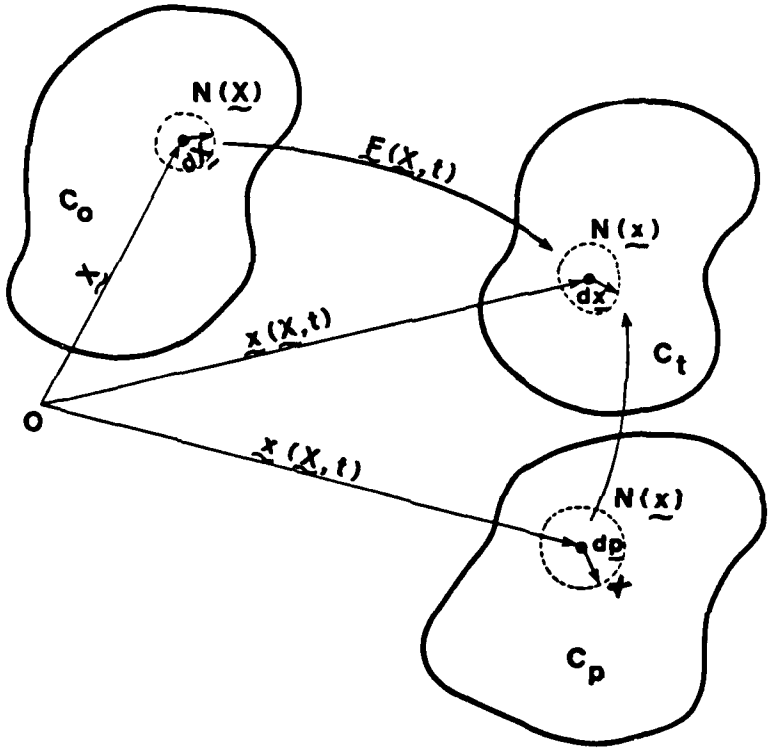


Figure 3.1 Popular configurations in finite elasto-plasticity.

3.1.1 *NEMAT-NASSER's Decomposition*

We first consider NEMAT-NASSER's additive decomposition which can be obtained by manipulation of a displacement field as follows [cf., NEMAT-NASSER [1979, p. 166]]:

$$\underset{\sim}{dx} - \underset{\sim}{dX} = (\underset{\sim}{F} - \underset{\sim}{I})\underset{\sim}{dX} = \underset{\sim}{du} \tag{3.3}$$

$$\underset{\sim}{dp} - \underset{\sim}{dX} = (\underset{\sim}{F}^{p} - \underset{\sim}{I})\underset{\sim}{dX} = \underset{\sim}{du}^{p} \tag{3.4}$$

If we demand that a single-valued displacement field must be realized in reaching the current configuration, we are led to a definition of an elastic deformation gradient given by

$$[\underset{\sim}{u}(X) + \underset{\sim}{du}] - [\underset{\sim}{u}(X) + \underset{\sim}{du}^{p}] = (\underset{\sim}{F}^{e} - \underset{\sim}{I})\underset{\sim}{dX} \tag{3.5}$$

We next use equations (3.3),(3.4) and (3.5) to eliminate $\underset{\sim}{du}$ and $\underset{\sim}{du}^{p}$ and obtain

$$\underset{\sim}{F} = \underset{\sim}{F}^{e} + \underset{\sim}{F}^{p} - \underset{\sim}{I} \tag{3.6}$$

In other words,

$$\underset{\sim}{F}^{e}\underset{\sim}{dX} = (\underset{\sim}{du} - \underset{\sim}{du}^{p} + \underset{\sim}{dX}) \tag{3.7}$$

and

$$\underset{\sim}{F}^{p}\underset{\sim}{dX} = \underset{\sim}{dp} = (\underset{\sim}{du}^{p} + \underset{\sim}{dX}) \tag{3.8}$$

Since the material derivative of x is defined by $\dot{\underset{\sim}{x}} = \partial \underset{\sim}{x}(\underset{\sim}{x},t)/\partial t = \dot{\underset{\sim}{u}}$,

$$\underset{\sim}{L} = \frac{\partial}{\partial \underset{\sim}{x}}\frac{\partial \underset{\sim}{u}}{\partial t} = \frac{\partial\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}\right)}{\partial t}\frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{x}} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} \tag{3.9}$$

and, from equation (3.6)

$$\dot{\underset{\sim}{F}} = \dot{\underset{\sim}{F}}^{e} + \dot{\underset{\sim}{F}}^{p} \tag{3.10}$$

We have

$$\underset{\sim}{L} = \dot{\underset{\sim}{F}}^{e}\underset{\sim}{F}^{-1} + \dot{\underset{\sim}{F}}^{p}\underset{\sim}{F}^{-1} = \underset{\sim}{L}^{e} + \underset{\sim}{L}^{p} \tag{3.11}$$

Finally, the symmetric part of $\underset{\sim}{L}$ is given by the sum,

$$\underset{\sim}{D} = \tfrac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T) = \tfrac{1}{2}(\underset{\sim}{L}^e + \underset{\sim}{L}^{eT}) + \tfrac{1}{2}(\underset{\sim}{L}^p + \underset{\sim}{L}^{pT}) \qquad (3.12)$$

$$= \underset{\sim}{D}^e + \underset{\sim}{D}^p$$

where $D^e = \underset{\sim}{F}^e\underset{\sim}{F}^{-1}|_{sym}$ $\underset{\sim}{D}^p = \underset{\sim}{F}^p\underset{\sim}{F}^{-1}|_{sym}$. (3.13)

and $|_{sym}$ denotes the symmetric part of the tensor.

3.1.2 *LEE's Decomposition*

Lee's decomposition starts "from the chain rule" [see LUBARDA and LEE [1981, p.4]:

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{p}} \frac{\partial \underset{\sim}{p}}{\partial \underset{\sim}{X}} = \bar{\underset{\sim}{F}}^e\underset{\sim}{F}^p \qquad (3.14)$$

Then, by using equations (3.11) and (3.14),

$$\underset{\sim}{L} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} = (\dot{\bar{\underset{\sim}{F}}}^e\underset{\sim}{F}^p + \bar{\underset{\sim}{F}}^e\dot{\underset{\sim}{F}}^p)\underset{\sim}{F}^{p-1}\bar{\underset{\sim}{F}}^{e-1}$$

$$= \dot{\bar{\underset{\sim}{F}}}^e\bar{\underset{\sim}{F}}^{e-1} + \bar{\underset{\sim}{F}}^e\dot{\underset{\sim}{F}}^p\underset{\sim}{F}^{p-1}\bar{\underset{\sim}{F}}^{e-1} \qquad (3.15)$$

$$= \bar{\underset{\sim}{L}}^e + \bar{\underset{\sim}{F}}^e\bar{\underset{\sim}{L}}^p\bar{\underset{\sim}{F}}^{e-1}$$

We recognize from (3.5) that the rate-of-strain measure $\underset{\sim}{D}$ could not be decomposed additively, as Lee noted. In metals, elastic strain rates may be small, and equation (3.12) can be approximated by

$$\underset{\sim}{L} \simeq \bar{\underset{\sim}{L}}^e + \bar{\underset{\sim}{L}}^p \quad \text{and} \quad \underset{\sim}{D} = \bar{\underset{\sim}{D}}^e + \bar{\underset{\sim}{D}}^p \qquad (3.16)$$

But this additive decomposition is, in general, invalid for the (perhaps unusual) case of finite elastic deformations.

Although we want a strain rate which is independent of the rotation effects, the previous decompositions do not render a full

stretch-dependent strain rate since, e.g., $\underset{\sim}{D}^P = \dot{\underset{\sim}{F}}^P \underset{\sim}{F}^{-1}|_{sym}$ and $\underset{\sim}{F}$ contains the rotation effect as $\underset{\sim}{F} = \underset{\sim}{RU} = \underset{\sim}{VR}$. Furthermore, the previous decompositions pose a question about the conceptual validity of the differential operation $d\underset{\sim}{p}/d\underset{\sim}{x}$. This fact may generate an overdetermined system of equations. The determinacy of a system of elasto-plastic equations will be discussed in a later section of this chapter after all the field equations and the constitutive equations are established through the theory of materials of type N.

To cure the difficulties mentioned above, we will search for new kinematical variables.

3.1.3 *Alternative Decomposition Methods*

Consider again a particle A in the neighborhood $N(\underset{\sim}{X})$ of $\underset{\sim}{X}$, in the reference configuration (see Fig. 3.2 or Fig. 3.3). The position vector of point $\underset{\sim}{A}$ relative to the origin of the fixed spatial reference frame is denoted

$$\overline{OA}_{C_0} = \underset{\sim}{X} + \Delta\underset{\sim}{X}$$

According to the polar decomposition theorem (see e.g. GURTIN [1981]), the deformation gradient $\underset{\sim}{F}$ can be represented as the composition,

$$\underset{\sim}{F} = \underset{\sim}{RU} = \underset{\sim}{VR} \tag{3.17}$$

where $\underset{\sim}{R}$ is a positive definite orthogonal rotation tensor, $\underset{\sim}{U}$ is the right stretch tensor of $\underset{\sim}{F}$, and $\underset{\sim}{V}$ is the left stretch tensor of $\underset{\sim}{F}$.

A) Right Polar Decomposition

The location of $\underset{\sim}{A}$ in the rotation free current configuration C_{t-}, as in Fig. 3.2, is

$$\overline{OA}_{C_{t-}} = \underset{\sim}{x} + \underset{\sim}{R}^T\Delta\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t) + \underset{\sim}{U}(\underset{\sim}{X},t)\Delta\underset{\sim}{X} + \underset{\sim}{W}_1(\underset{\sim}{X},\ t,\ \Delta\underset{\sim}{X}) \tag{3.18}$$

where

$$\lim_{\|\Delta\underset{\sim}{X}\|\to 0} \frac{1}{\|\Delta\underset{\sim}{X}\|} \|\underset{\sim}{W}_1(\underset{\sim}{X},\ t,\ \Delta\underset{\sim}{X})\| = 0 \tag{3.19}$$

Now let us consider an intermediate configuration C_{p-} which may correspond to a "rotation free" state at the initial stress level $\underset{\sim}{\sigma}_0$. Then,

$$\overline{OA}_{C_{p-}} = \underset{\sim}{x}(\underset{\sim}{X},t) + \underset{\sim}{U}^p(\underset{\sim}{X},t)\Delta\underset{\sim}{X} + \underset{\sim}{W}_2(\underset{\sim}{X},\ t,\ \Delta\underset{\sim}{X}) \tag{3.20}$$

and $\underset{\sim}{W}_2$ has the same asymptotic behavior with respect to $\|\Delta\underset{\sim}{X}\|$ as does $\underset{\sim}{W}_1$.

We next introduce a second order tensor $\underset{\sim}{U}^e$ which represents the elastic stretch tensor and is defined by

$$\begin{aligned}\underset{\sim}{U}^e(\underset{\sim}{X},t)\Delta\underset{\sim}{X} - \Delta\underset{\sim}{X} &= \overline{OA}_{C_{t-}} - \overline{OA}_{C_{p-}} \\ &= [\underset{\sim}{U}(\underset{\sim}{X},t) - \underset{\sim}{U}^p(\underset{\sim}{X},t)]\Delta\underset{\sim}{X} \\ &\quad + \underset{\sim}{W}_1(\underset{\sim}{X},\ t,\ \Delta\underset{\sim}{X}) - \underset{\sim}{W}_2(\underset{\sim}{X},\ t,\ \Delta\underset{\sim}{X})\end{aligned} \tag{3.21}$$

Thus, in the limit as $|\Delta\underset{\sim}{X}| \to 0$, we have

$$\underset{\sim}{U} = \underset{\sim}{U}^e + \underset{\sim}{U}^p - \underset{\sim}{1} \tag{3.22}$$

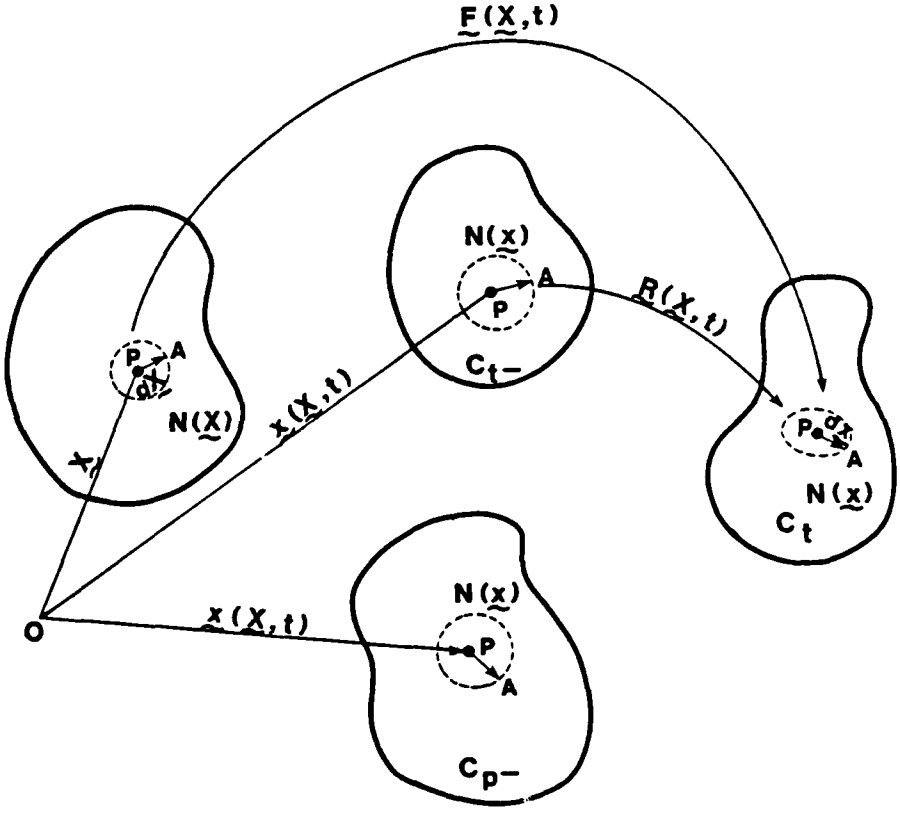


Figure 3.2 Configurations by the right polar decomposition.

and the relations (3.17) and (3.11) give

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{R}\underset{\sim}{U}^{e} + \underset{\sim}{R}\underset{\sim}{U}^{p} - \underset{\sim}{R} \tag{3.23}$$

$$\underset{\sim}{L} = \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1}$$

$$= \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T} + \underset{\sim}{R}(\dot{\underset{\sim}{U}}^{e}\underset{\sim}{U}^{-1} + \dot{\underset{\sim}{U}}^{p}\underset{\sim}{U}^{-1})\underset{\sim}{R}^{T} \tag{3.24}$$

Therefore, $\underset{\sim}{D}^{e}$ and $\underset{\sim}{D}^{p}$ can be obtained as

$$\underset{\sim}{D}^{e} = \underset{\sim}{R}\,\dot{\underset{\sim}{U}}^{e}\,\underset{\sim}{U}^{-1}\,\underset{\sim}{R}^{T}\big|_{sym} \tag{3.25}$$

$$\underset{\sim}{D}^{p} = \underset{\sim}{R}\,\dot{\underset{\sim}{U}}^{p}\,\underset{\sim}{U}^{-1}\,\underset{\sim}{R}^{T}\big|_{sym} \tag{3.26}$$

B) Left Polar Decomposition

The left polar decomposition cannot be constructed as easily and simply as the right, but a formal manipulation of the key tensors is still possible. In this case, the material is first rotated as in Fig. 3.3. Therefore, the location of $\underset{\sim}{A}$ in the rotated reference configuration C_{0+} is

$$OA_{C} = \underset{\sim}{X} + \underset{\sim}{R}\,\Delta\underset{\sim}{X} \tag{3.27}$$

whereas the locations of $\underset{\sim}{A}$ in the current configuration C_t and in the rotated stress-free configuration C_{p+} are

$$\overline{OA}_{C_t} = \underset{\sim}{x} + \Delta\underset{\sim}{x}$$

$$= \underset{\sim}{x}(X,t) + \underset{\sim}{V}(X,t)\underset{\sim}{R}\Delta\underset{\sim}{X} + \underset{\sim}{W}_3(X,\ t,\ \Delta\underset{\sim}{X}) \tag{3.28}$$

and

$$\overline{OA}_{C_{p+}} = \underset{\sim}{x}(X,t) + \underset{\sim}{V}^{p}(X,\ t)\underset{\sim}{R}\Delta\underset{\sim}{X} + \underset{\sim}{W}_4(X,\ t\ \Delta\underset{\sim}{X}) \tag{3.29}$$

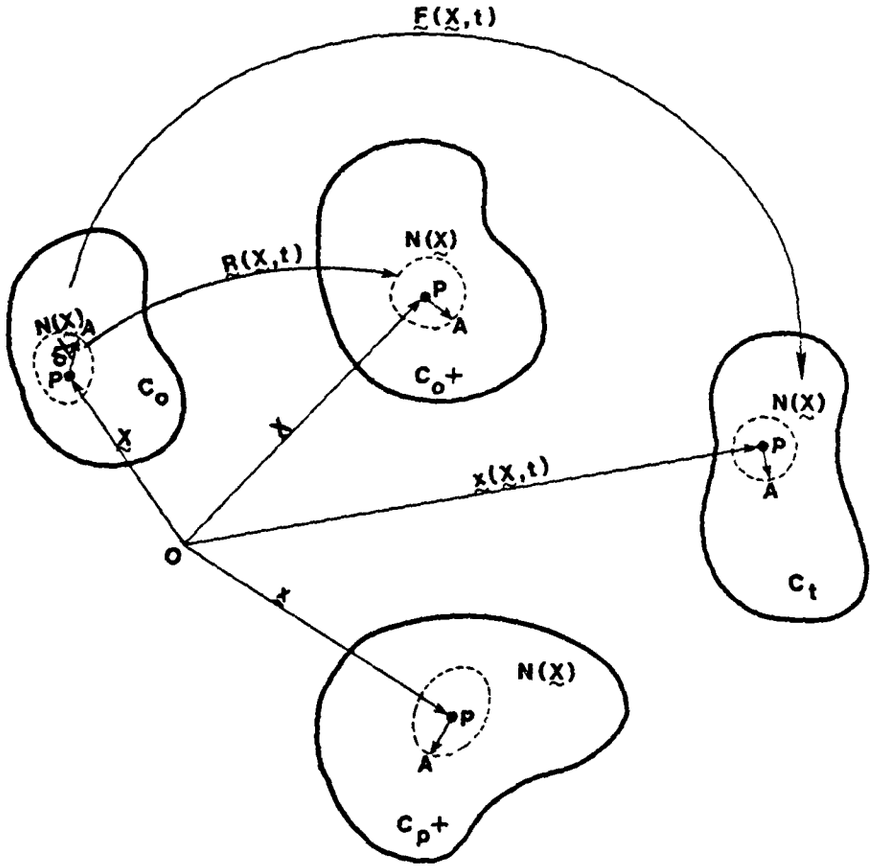


Figure 3.3 Configurations by the left polar decomposition.

Here $\underset{\sim}{W}_3$ and $\underset{\sim}{W}_4$ have the same asymptotic property as do $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$.

In a way similar to (3.21), a second-order elastic (left) stretch tensor $\underset{\sim}{V}^e$ is defined by

$$\begin{aligned}\underset{\sim}{V}^e(X, t)R\Delta\underset{\sim}{X} - R\Delta\underset{\sim}{X} &= \overline{OA}_{C_t} - \overline{OA}_{C_{p+}} \\ &= [\underset{\sim}{V}(X, t) - \underset{\sim}{V}^p(x, t)]R\Delta\underset{\sim}{X} \\ &+ \underset{\sim}{W}_3(X, t, \Delta\underset{\sim}{X}) - \underset{\sim}{W}_4(X, t, \Delta\underset{\sim}{X})\end{aligned} \tag{3.30}$$

And $\|\Delta\underset{\sim}{X}\| \to 0$ gives

$$\underset{\sim}{V} = \underset{\sim}{V}^e + \underset{\sim}{V}^p - \underset{\sim}{1} \tag{3.31}$$

or

$$\underset{\sim}{F} = \underset{\sim}{V}\underset{\sim}{R} = \underset{\sim}{V}^e \underset{\sim}{R} + \underset{\sim}{V}^p \underset{\sim}{R} - \underset{\sim}{R} \tag{3.32}$$

Finally, from the definitions of $\underset{\sim}{L}$ and $\underset{\sim}{D}$,

$$\begin{aligned}\underset{\sim}{L} &= \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1} \\ &= \dot{\underset{\sim}{V}}^e \underset{\sim}{V}^{-1} + \dot{\underset{\sim}{V}}^p \underset{\sim}{V}^{-1} + \underset{\sim}{V}\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^T \underset{\sim}{V}^{-1} \\ &\equiv \dot{\underset{\sim}{V}}^e \underset{\sim}{V}^{-1} + \underset{\sim}{V}^p \underset{\sim}{V}^{-1} + \underset{\sim}{g}_1(\underset{\sim}{V}^e, \underset{\sim}{R}) + \underset{\sim}{g}_2(\underset{\sim}{V}^p, \underset{\sim}{R})\end{aligned} \tag{3.33}$$

where, for example,

$$\underset{\sim}{g}_1(\underset{\sim}{V}^e\underset{\sim}{R}) = \underset{\sim}{V}^e\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^T\underset{\sim}{V}^{-1} \quad \text{and} \quad \underset{\sim}{g}_2(\underset{\sim}{V}^p, \underset{\sim}{R}) = \underset{\sim}{V}^p\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^T\underset{\sim}{V}^{-1} - \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^T\underset{\sim}{V}^{-1} \tag{3.34}$$

Then one could define,

$$\underset{\sim}{D}^e = (\dot{\underset{\sim}{V}}^e \underset{\sim}{V}^{-1} + \underset{\sim}{g}_1(\underset{\sim}{V}^e, \underset{\sim}{R}))\big|_{sym} \tag{3.35}$$

and

$$\underset{\sim}{D}^P = (\dot{\underset{\sim}{V}}^e \underset{\sim}{V}^{-1} + \underset{\sim}{g}_2(\underset{\sim}{V}^P, \underset{\sim}{R}))\big|_{sym} \tag{3.36}$$

It is noted here that unlike NEMAT-NASSER [1979, 1982] or LEE [1969, 1981], neither $\underset{\sim}{V}^e$ nor $\underset{\sim}{V}^P$ need be a function of the gradient of an "elastic" or "plastic" displacement or position vector field. However, if $\underset{\sim}{u}$ is the particle displacement field, we always have,

$$\underset{\sim}{F} = \underset{\sim}{1} + \nabla\underset{\sim}{u} \tag{3.37}$$

It has recently been pointed out to us that our decomposition through (3.17) is similar to that proposed by SIMO and MARSDEN [1984].

3.2 Materials of Type N

The thermomechanical behavior of the body is governed by the principles of conservations of mass and energy, balances of the linear and the angular momenta, and the law of entropy production. Local forms of these principles can be written as follows:

·Conservation of Mass

$$\rho \det \underset{\sim}{F} = \rho_0 \tag{3.38}$$

·Balance of Linear and Angular Momenta

$$\text{div}\, \underset{\sim}{\sigma} + \rho\, \underset{\sim}{b} = \rho \ddot{\underset{\sim}{x}} \tag{3.39}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \tag{3.40}$$

·Conservation of Energy

$$\rho \dot{\epsilon} = (\underset{\sim}{\sigma} : \underset{\sim}{L}) - \text{div}\, \underset{\sim}{q} + \rho r \tag{3.41}$$

·Clausius - Duhem Inequality

$$\rho\dot{\eta} + \operatorname{div}\frac{\underline{q}}{\theta} - \frac{r}{\theta} \geq 0 \tag{3.42}$$

Here ρ is the mass density, ρ_o the mass density in the reference configuration, $\underset{\sim}{\sigma}$ is the Cauchy Stress, $\underset{\sim}{b}$ the body force per unit mass, ϵ the specific internal energy, $\underset{\sim}{q}$ the heat flux vector, r the heat supply per unit mass per unit time, η the specific entropy, and θ the absolute temperature.

It is customary to introduce the free energy density function instead of the internal energy density ϵ by

$$\phi = \epsilon - \eta\theta \tag{3.43}$$

Now we are at the place to define a general class of materials which satisfies certain properties.

Let us first introduce sets $S \subset \mathbb{R}^3 \times \mathbb{R}^3$, $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ and $\bar{W} = S \times A$.

A material is said to be of **Type N** if and only if it is characterized by constitutive equations,

$$\begin{aligned} \phi &= \Phi\ (\underset{\sim}{E},\theta,\underset{\sim}{g},\underset{\sim}{\alpha}) \\ \underset{\sim}{\sigma} &= \underset{\sim}{\Sigma}\ (\underset{\sim}{E},\theta,\underset{\sim}{g},\underset{\sim}{\alpha}) \\ \eta &= N\ (\underset{\sim}{E},\theta,\underset{\sim}{g},\underset{\sim}{\alpha}) \\ \underset{\sim}{q} &= \underset{\sim}{Q}\ (\underset{\sim}{E},\theta,\underset{\sim}{g},\underset{\sim}{\alpha}) \end{aligned} \tag{3.44}$$

and there exists a potential $\psi:\bar{W} \to (-\infty,\infty)$. Which has a nonempty generalized subdifferential such that

$$(\underset{\sim}{\dot{P}},-\underset{\sim}{\dot{\alpha}}) \in \bar{\partial}\psi(\underset{\sim}{\sigma},\underset{\sim}{A}),\ \forall\ (\underset{\sim}{\sigma},\underset{\sim}{A}) \in \bar{W} \tag{3.45}$$

Here g = grad θ and is a tensor referred to as an internal state variable. The elastic strain measure E and the plastic strain measure $\underset{\sim}{P}$ were used without specific relations with the kinematical variables. We will investigate these for all four cases which were introduced in Section 3.1.

From the result (2.13), the relation (3.45) implies the inequality

$$\langle \dot{\underset{\sim}{P}}, \underset{\sim}{\sigma}^* \rangle_S + \langle -\dot{\underset{\sim}{\alpha}}, \underset{\sim}{A}^* \rangle_A \leq D\uparrow\psi((\underset{\sim}{\sigma}, \underset{\sim}{A}); (\underset{\sim}{\sigma}^*, \underset{\sim}{A}^*)) \quad \forall (\underset{\sim}{\sigma}^*, \underset{\sim}{A}^*) \in \bar{W} \tag{3.46}$$

Where $\langle \cdot, \cdot \rangle_S$ and $\langle \cdot, \cdot \rangle_A$ denote duality paring on S* x S and A* x A, respectively.

Geometrically, we have the normality (hence, Type N) condition,

$$((\dot{\underset{\sim}{P}}, -\dot{\underset{\sim}{\alpha}}), -1) \in N_{epi\psi}[(\underset{\sim}{\sigma}, \underset{\sim}{A}), \psi(\underset{\sim}{\sigma}, \underset{\sim}{A})] \tag{3.47}$$

This relation is illustrated graphically in Fig. 3.4 with $U = (\underset{\sim}{\sigma}, \underset{\sim}{A})$ and $F = \psi$.

Furthermore, if the potential ψ is differentiable at $(\underset{\sim}{\sigma}, \underset{\sim}{A})$ then $N_{epi\psi}$ at $(\underset{\sim}{\sigma}, \underset{\sim}{A})$ has a single element. So,

$$(\dot{\underset{\sim}{P}}, -\dot{\underset{\sim}{\alpha}}) = \frac{\partial \psi}{\partial (\underset{\sim}{\sigma}, \underset{\sim}{A})}$$

or

$$\left.\begin{aligned} \dot{\underset{\sim}{P}} &= \frac{\partial \psi}{\partial \underset{\sim}{\sigma}} \\ \text{and} \\ \dot{\underset{\sim}{\alpha}} &= -\frac{\partial \psi}{\partial \underset{\sim}{A}} \end{aligned}\right\} \tag{3.48}$$

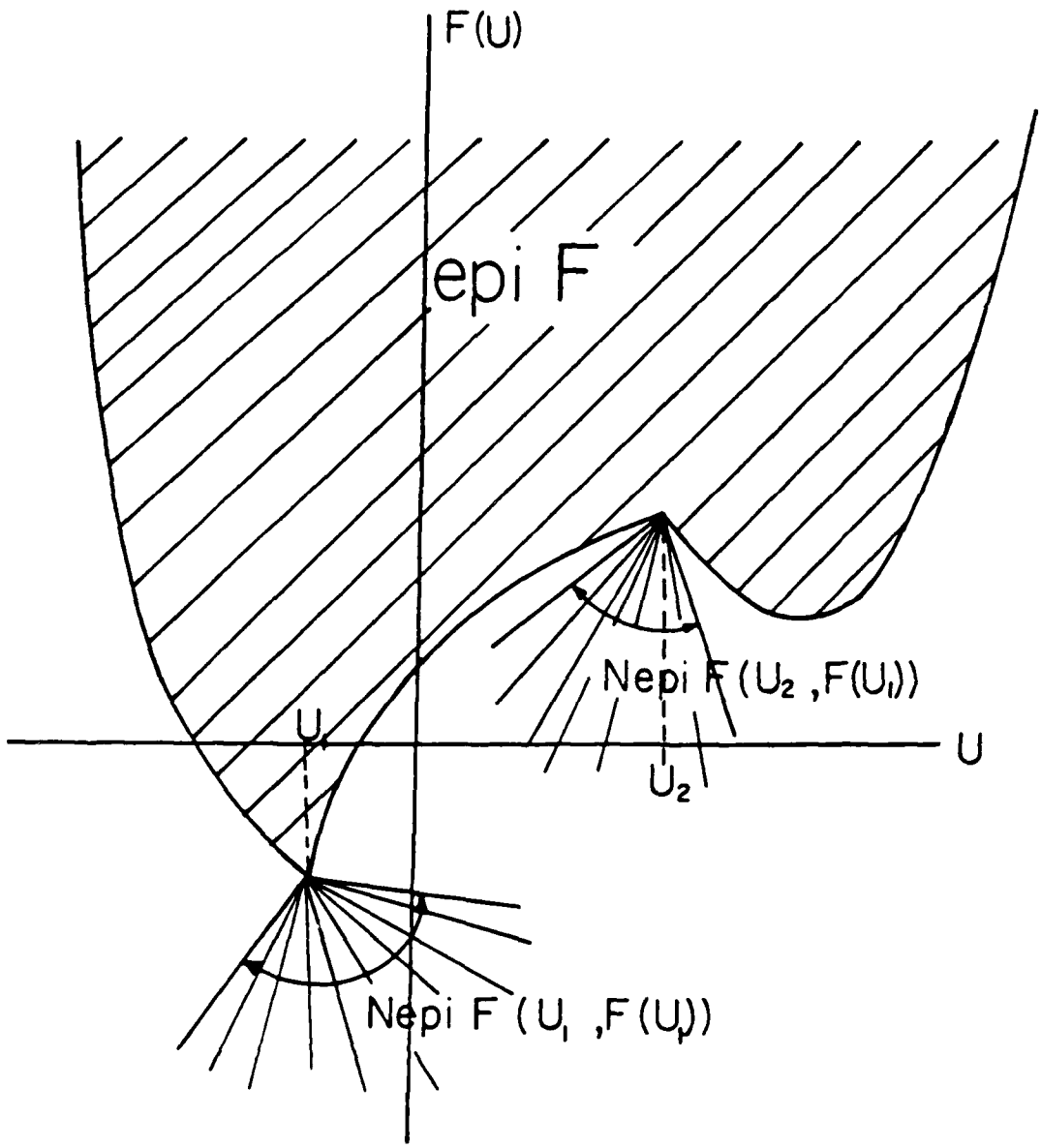


Figure 3.4 Normal cones at u_1 in convex neighborhood and u_2 in non-convex neighborhood.

We remark that if ψ is convex and l.s.c., the materials of Type N reduce to the generalized materials introduced by HALPHEN and NGUYEN [1975] that is, from Theorem 2.2,

$$\bar{\partial}\psi(\underset{\sim}{\sigma},\underset{\sim}{A}) = \partial\psi(\underset{\sim}{\sigma},\underset{\sim}{A})$$

$$\Rightarrow \quad \langle(\dot{\underset{\sim}{P}},-\dot{\underset{\sim}{\alpha}}),\ (\underset{\sim}{\sigma}^*,\underset{\sim}{A}^*)-(\underset{\sim}{\sigma},\underset{\sim}{A})\rangle \leq \psi(\underset{\sim}{\sigma}^*,\underset{\sim}{A}^*)-\psi(\underset{\sim}{\sigma},\underset{\sim}{A}^*) \forall\, (\underset{\sim}{\sigma}^*,\underset{\sim}{A}^*) \in \bar{W}$$

or

$$\langle \dot{\underset{\sim}{P}},\underset{\sim}{\sigma}^*-\underset{\sim}{\sigma}\rangle + \langle -\dot{\underset{\sim}{\alpha}},\underset{\sim}{A}^*-\underset{\sim}{A}\rangle \leq \psi(\underset{\sim}{\sigma}^*,\underset{\sim}{A}^*)-\psi(\underset{\sim}{\sigma},\underset{\sim}{A}) \quad \forall\, (\underset{\sim}{\sigma}^*,\underset{\sim}{A}^*) \in \bar{W} \tag{3.49}$$

Eliminating r between (3.41) and(3.42), and taking into account (3.43), we obtain

$$\underset{\sim}{\sigma}:\underset{\sim}{L} - \rho\dot{\phi} - \rho\dot{\eta}n - \frac{1}{\theta}\,\underset{\sim}{q}\cdot\nabla\theta \geq 0 \tag{3.50}$$

Assume that the map: $(\underset{\sim}{E}, \theta, \underset{\sim}{g}, \underset{\sim}{\alpha}) \rightarrow \phi(\cdot,\cdot,\cdot,\cdot)$ is C^1 in each argument. Then the rate of change of the free energy can be expanded by

$$\dot{\phi} = \frac{\partial\phi}{\partial\underset{\sim}{E}} : \dot{\underset{\sim}{E}} + \frac{\partial\phi}{\partial\theta}\,\dot{\theta} + \frac{\partial\phi}{\partial\underset{\sim}{g}} \cdot \dot{\underset{\sim}{g}} + \frac{\partial\phi}{\partial\underset{\sim}{\alpha}} : \dot{\underset{\sim}{\alpha}} \tag{3.51}$$

Where the inner product notations ": and ." for tensors (2nd order) and vectors mean that

$$\underset{\sim}{A}:\underset{\sim}{B} = \mathrm{tr}(\underset{\sim}{A}^T\underset{\sim}{B}) = \sum_{i,j} A_{ij}\,B_{ij} \quad \text{and} \quad \underset{\sim}{a}.\underset{\sim}{b} = \sum a_i b_i$$

Putting (3.51) into 3.50) gives

$$\underset{\sim}{\sigma}:\underset{\sim}{L} - \rho\frac{\partial\phi}{\partial\underset{\sim}{E}} : \dot{\underset{\sim}{E}} - \rho\left(\frac{\partial\phi}{\partial\theta} + \eta\right)\dot{\theta} - \rho\,\frac{\partial\phi}{\partial\underset{\sim}{g}}\cdot\dot{g} - \frac{\partial\phi}{\partial\underset{\sim}{\alpha}} : \dot{\underset{\sim}{\alpha}} - \frac{1}{\theta}\,\underset{\sim}{q}\cdot\nabla\theta \geq 0 \tag{3.52}$$

Ncticing that none of the constitutive variables in the assumption of materials of Type N are functions of $\dot{\theta}$ and $\dot{\underset{\sim}{g}}$, we can vary the variables $\dot{\theta}$, $\dot{\underset{\sim}{g}}$ without changing the other variables. Then we will obtain the following equalities

$$\eta = N(\underset{\sim}{E},\theta,\underset{\sim}{g},\underset{\sim}{\alpha}) = -\frac{\partial\phi}{\partial\theta} \tag{3.53}$$

$$\underset{\sim}{0} = \frac{\partial\phi}{\partial\underset{\sim}{g}} \tag{3.54}$$

and ensuing inequality

$$\underset{\sim}{\sigma}:\underset{\sim}{L} - \rho\frac{\partial\phi}{\partial\underset{\sim}{E}} : \dot{\underset{\sim}{E}} + \underset{\sim}{A} : \dot{\underset{\sim}{\alpha}} - \frac{1}{\theta}\,\underset{\sim}{q}\cdot\nabla\theta \geq 0 \tag{3.55}$$

where $\underset{\sim}{A} = \frac{\partial\phi}{\partial\underset{\sim}{\alpha}}$

Equation (3.54) implies that ϕ is independent of the temperature gradient $\nabla\theta$ and so is the specific entropy η.

We notice here that we need to know the relations among $\underset{\sim}{L}$, $\underset{\sim}{E}$ and $\underset{\sim}{P}$ to obtain more restrictions from the inequality (3.55).

3.2.1 *Case I*

We first apply the kinematical definition of NEMAT-NASSER, i.e., from (3.13)

$$\underset{\sim}{E}_I = \int_0^t \dot{\underset{\sim}{F}}^e\underset{\sim}{F}^{-1}\Big|_{sym}\,dt \tag{3.56}$$

$$\underset{\sim}{P}_I = \int_0^t \dot{\underset{\sim}{F}}^p\underset{\sim}{F}^{-1}\Big|_{sym}\,dt \tag{3.57}$$

Then

$$\underset{\sim}{\sigma}:\underset{\sim}{L} = \underset{\sim}{\sigma}:\underset{\sim}{D} = \underset{\sim}{\sigma}:(\dot{\underset{\sim}{E}}_I + \dot{\underset{\sim}{P}}_I)$$

and the inequality

$$(\underset{\sim}{\sigma} - \rho \frac{\partial \phi}{\partial \underset{\sim}{E}}) : \dot{\underset{\sim}{E}}_I + \underset{\sim}{\sigma}:\dot{\underset{\sim}{P}}_I + \underset{\sim}{A} : \dot{\underset{\sim}{\alpha}} - \frac{1}{\theta} \underset{\sim}{q} \cdot \nabla\theta \geq 0 \qquad (3.58)$$

gives one more restriction

$$\underset{\sim}{\sigma} = \rho \frac{\partial \phi}{\partial \underset{\sim}{E}_I} \qquad (3.59)$$

and inequality

$$\underset{\sim}{\sigma}:\dot{\underset{\sim}{P}}_I + \underset{\sim}{A}:\dot{\underset{\sim}{\alpha}} - \frac{1}{\theta} \underset{\sim}{q}\cdot\nabla\theta \geq 0 \qquad (3.60)$$

where the thermodynamic force A, which is designed to be conjugate variable to the internal state variables, is defined by

$$\underset{\sim}{A} = - \frac{\partial \phi}{\partial \underset{\sim}{\alpha}} \qquad (3.61)$$

3.2.2 *Case II*

The result (3.15) of LEE's decomposition gives

$$\underset{\sim}{E}_{II} = \int_0^t \dot{\underset{\sim}{F}}^e \, \underset{\sim}{F}^{e-1} \Big|_{sym} dt \qquad (3.62)$$

$$\underset{\sim}{P}_{II} = \int_0^t \underset{\sim}{F}^e \, \dot{\underset{\sim}{F}}^p \, \underset{\sim}{F}^{p-1} \, \underset{\sim}{F}^{e-1} \Big|_{sym} dt \qquad (3.63)$$

In a similar way, we get

$$\underset{\sim}{\sigma} = \rho \frac{\partial \phi}{\partial \underset{\sim}{E}_{II}} \qquad (3.64)$$

and

$$\underset{\sim}{\sigma}:\dot{\underset{\sim}{P}}_{II} + \underset{\sim}{A}:\dot{\underset{\sim}{\alpha}} - \frac{1}{\theta}\,\underset{\sim}{q}\cdot\nabla\theta \geq 0 \tag{3.65}$$

REMARK: The definition of plastic strain (3.63) may lead to some difficulties, as noted by NEMAT-NASSER [1979, 1981]. Since (3.63) contains tensor $\bar{\underset{\sim}{F}}^{e}$ which represents the elastic deformation, there arises the question as to whether or not $\underset{\sim}{P}_{II}$ really represents the plastic deformation. Furthermore, the definition $\bar{\underset{\sim}{F}}^{e}$ which is used to define the elastic strain (3.62) contains effects from the intermediate plastic configuration Cp as in Fig. 3.1. Here we assume that we can vary the variables $\dot{\underset{\sim}{E}}$ and $\dot{\underset{\sim}{P}}$ independently. When we have a deformation where significant couping between $\underset{\sim}{E}$ and $\underset{\sim}{P}$ exists, (3.64) and (3.65) may be invalid. □

3.2.3 *Case III*

From (3.25) and (3.26) we define strains by

$$\underset{\sim}{E}_{III} = \int_0^t \dot{\underset{\sim}{U}}^{e}\underset{\sim}{U}^{-1}\,|_{sym}\,dt \tag{3.66}$$

$$\underset{\sim}{P}_{III} = \int_0^t \dot{\underset{\sim}{U}}^{p}\underset{\sim}{U}^{-1}\,|_{sym}\,dt \tag{3.67}$$

then

$$\underset{\sim}{D} = \underset{\sim}{R}\,\dot{\underset{\sim}{E}}_{III}\underset{\sim}{R}^{T} + \underset{\sim}{R}\,\dot{\underset{\sim}{P}}_{III}\underset{\sim}{R}^{T} \tag{3.68}$$

Substituting (3.68) into (3.55) gives

$$(\underset{\sim}{R}^T \underset{\sim}{\sigma}\, \underset{\sim}{R} - \rho \frac{\partial \phi}{\partial \underset{\sim}{E}_{III}}) : \dot{\underset{\sim}{E}}_{III} + \underset{\sim}{R}^T \underset{\sim}{\sigma}\, \underset{\sim}{R} : \dot{\underset{\sim}{P}}_{III}$$

$$+ \underset{\sim}{A} : \dot{\underset{\sim}{\alpha}} - \frac{1}{\theta} \underset{\sim}{q} \cdot \nabla\theta \geq 0 \qquad (3.69)$$

Here we rename $\underset{\sim}{R}^T \underset{\sim}{\sigma} \underset{\sim}{R} = \underset{\sim}{S}$ and we call $\underset{\sim}{S}$ Dienes stress since it has appeared in the the work by DIENS [1979].

By the way, from (3.69) we get

$$\underset{\sim}{S} = \rho \frac{\partial \phi}{\partial \underset{\sim}{E}_{III}} \qquad (3.70)$$

and

$$\underset{\sim}{S} : \dot{\underset{\sim}{P}}_{III} + \underset{\sim}{A} : \dot{\underset{\sim}{\alpha}} - \frac{1}{\theta} \underset{\sim}{q} \cdot \nabla \theta \geq 0 \qquad (3.71)$$

REMARK: Kinematically, Dienes stress $\underset{\sim}{S}$ has a special meaning. Since $\underset{\sim}{S}$ is defined as a reverse rotation of Cauchy stress $\underset{\sim}{\sigma}$, it is the stress in the configuration C_t- in Fig. 3.2. The configuration C_t- is designed to be a rotation-free configuration. Therefore, variables $\underset{\sim}{U}$, $\underset{\sim}{S}$ and etc. in C_t- are rotation free and we are able to obtain a materially objective stress rate. This issue has prompted many discussions of appropriate measures of stress rate (see, e.g. DIENES [1979], NAGTEGAAL and DeJONG [1982], ATLURI [1980, 1983] and LEE and BERTHEIMER[1983]). □

From the definition

$$\underset{\sim}{S} = \underset{\sim}{R}^T \underset{\sim}{\sigma}\, \underset{\sim}{R} \qquad (3.72)$$

we have

$$\dot{\underset{\sim}{S}} = \dot{\underset{\sim}{R}}^T \underset{\sim}{\sigma}\, \underset{\sim}{R} + \underset{\sim}{R}^T \dot{\underset{\sim}{\sigma}}\, \underset{\sim}{R} + \underset{\sim}{R}^T \underset{\sim}{\sigma}\, \dot{\underset{\sim}{R}}$$

$$= \underset{\sim}{R}^T (\dot{\underset{\sim}{\sigma}} - \dot{\underset{\sim}{R}}\underset{\sim}{R}^T \underset{\sim}{\sigma} + \underset{\sim}{\sigma}\, \dot{\underset{\sim}{R}}\underset{\sim}{R}^T) \underset{\sim}{R} \tag{3.73}$$

since

$$\underset{\sim}{R}\underset{\sim}{R}^T = \underset{\sim}{1} \Rightarrow \dot{\underset{\sim}{R}}\underset{\sim}{R}^T = - \underset{\sim}{R}\dot{\underset{\sim}{R}}^T \tag{3.74}$$

Then an objective rate of the Cauchy stress $\underset{\sim}{\sigma}$ is

$$\overset{\nabla}{\underset{\sim}{\sigma}} \equiv \underset{\sim}{R}\, \dot{\underset{\sim}{S}}\, \underset{\sim}{R}^T = \dot{\underset{\sim}{\sigma}} - \underset{\sim}{\omega}\, \underset{\sim}{\sigma} + \underset{\sim}{\sigma}\, \underset{\sim}{\omega} \tag{3.75}$$

where $\underset{\sim}{\omega} = \dot{\underset{\sim}{R}}\, \underset{\sim}{R}^T$

Notice that $\overset{\nabla}{\underset{\sim}{\sigma}}$ is a stress rate in the configuration C_t and $\dot{\underset{\sim}{S}}$ is a stress rate in C_t in the Fig. 3.2.

3.2.4 *Case IV*

From (3.35) and (3.36), we could define a set of strain measures,

$$\underset{\sim}{E}_{IV} = \int_o^t (\dot{\underset{\sim}{V}}^e \underset{\sim}{V}^{-1} + \underset{\sim}{g}_1(\underset{\sim}{V}^e, \underset{\sim}{R}))\,|_{sym}\, dt \tag{3.76}$$

$$\underset{\sim}{E}_{IV} = \int_o^t (\dot{\underset{\sim}{V}}^P \underset{\sim}{V}^{-1} + \underset{\sim}{g}_2(\underset{\sim}{V}^P, \underset{\sim}{R}))\,|_{sym}\, dt \tag{3.77}$$

Then, from (3.55), we get

$$\underset{\sim}{\sigma} = \rho \frac{\partial \phi}{\partial \underset{\sim}{E}_{IV}} \tag{3.78}$$

and

$$\underset{\sim}{\sigma} : \dot{\underset{\sim}{P}}_{III} + \underset{\sim}{A} : \dot{\underset{\sim}{\alpha}} - \frac{1}{\theta}\, \underset{\sim}{q} \cdot \nabla\theta \geq 0 \tag{3.79}$$

3.3 Evaluations of Various Strain Measures

Up to now, four sets of system of equations according to four kinematical definitions have been proposed. In this section we evaluate all of these cases and choose a strain measure which is both physically sound and convenient.

3.3.1 *Determinacy of Governing Equations*

Ignoring thermal effects explicitly, we again list the following equations:

i) Field equations

$$\rho \det \underset{\sim}{F} = \rho_o \tag{3.80}$$

$$\operatorname{div} \underset{\sim}{\sigma} + \rho \underset{\sim}{b} = \rho \ddot{\underset{\sim}{\chi}}, \quad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \tag{3.81}$$

ii) Constitutive equations

From (3.48), the plastic constitutive equations

$$\dot{\underset{\sim}{P}} = \frac{\partial \psi}{\partial \underset{\sim}{\sigma}} \tag{3.82}$$

$$\dot{\underset{\sim}{\alpha}} = -\frac{\partial \psi}{\partial \underset{\sim}{A}} \tag{3.83}$$

and the elastic constitative equation (one among (3.59),(3.64), (3.70) and (3.78))

$$\underset{\sim}{\sigma} (\text{or } \underset{\sim}{S}) = \rho \frac{\partial \phi}{\partial \underset{\sim}{E}} \tag{3.84}$$

and definition

$$\underset{\sim}{A} = -\frac{\partial \phi}{\partial \underset{\sim}{\alpha}}, \tag{3.85}$$

iii) Kinematical relations

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} \tag{3.86}$$

Case I:

$$\begin{cases} \underset{\sim}{F} = \underset{\sim}{F}^e + \underset{\sim}{F}^p - \underset{\sim}{1} & (3.87) \\ \underset{\sim}{F}^p = \frac{\partial p}{\partial \underset{\sim}{X}} & (3.88) \\ (3.56) \\ (3.57) \end{cases}$$

Case II:

$$\begin{cases} \underset{\sim}{F} = \bar{\underset{\sim}{F}}^e \, \underset{\sim}{F}^p & (3.89) \\ (3.87) \\ (3.62) \\ (3.63) \end{cases}$$

Case III:

$$\begin{cases} \underset{\sim}{F} = \underset{\sim}{R} \, \underset{\sim}{U} & (3.90) \\ \underset{\sim}{U} = \underset{\sim}{U}^e + \underset{\sim}{U}^p - \underset{\sim}{1} & (3.91) \\ (3.66) \\ (3.67) \end{cases}$$

Case IV:

$$\begin{cases} \underset{\sim}{F} = \underset{\sim}{V} \, \underset{\sim}{R} & (3.92) \\ \underset{\sim}{V} = \underset{\sim}{V}^e + \underset{\sim}{V}^p - \underset{\sim}{1} & (3.93) \\ (3.76) \\ (3.77) \end{cases}$$

The total number of unknowns and the total number of equations are listed in Tables 3.1 and 3.2 for all four cases.

The first and the second cases give 3 more equations than unknowns, i.e., overdetermined systems of equations are obtained.

Generally, we cannot solve an over determined system of equations like these. So we must find the way to eliminate the extra 3 equations. It is noted that the 3 extra equations result from the introduction of the variable $\underset{\sim}{P}$ which may not be a single valued function. But in the first two decompositions, we have assumed that the operation $d\underset{\sim}{p}/d\underset{\sim}{x}$ is valid, and this implies a compatibility condition is fulfilled.

The third and the fourth decompositions furnish well determined systems of equations. But the fourth case becomes very complicated since the rotation effects are not explicitly removed.

In the following section, the actual differences of the previously mentioned strain measures in the one dimensional homogeneous deformation are analyzed.

3.3.2 *Calculations of Elastic Strains*

We now consider a one-dimensional case of homogeneous deformation. The deformation gradients then reduce to stretch tensors (there are no rotations). In this case, the Nemat-Nasser decomposition reduces to the same representation as our right and

Table 3.1 Number of Unknowns

Variables	Case 1	Case 2	Case 3	Case 4
ρ	1	1	1	1
$\underset{\sim}{F}$	9	9	9	9
$\underset{\sim}{\sigma}$	6	6	6	6
$\underset{\sim}{P}$	6	6	6	6
$\underset{\sim}{E}$	6	6	6	6
$\underset{\sim}{\alpha}$	9	9	9	9
A	9	9	9	9
$\underset{\sim}{F}^e$	9			
$\underset{\sim}{F}^p$	9	9		
$\underset{\sim}{F}^e$		9		
$\underset{\sim}{U}^e$			6	
$\underset{\sim}{U}^p$			6	
$\underset{\sim}{V}^e$				6
$\underset{\sim}{V}^p$				6
$\underset{\sim}{x}$	3	3	3	3
$\underset{\sim}{P}$	3	3		
$\underset{\sim}{R}$			3	3
$\underset{\sim}{U}$			6	6
Total	70	70	70	70

Table 3.2 Number of Equations

Equations	Case 1	Case 2	Case 3	Case 4
3.80	1	1	1	1
3.81	3	3	3	3
3.82	6	6	6	6
3.83	9	9	9	9
3.84	6	6	6	6
3.85	9	9	9	9
3.86	9	9	9	9
3.87	9	9		
3.88	9			
3.56	6			
3.57	6			
3.89		9		
3.62		6		
3.63		6		
3.90			9	
3.91			6	
3.66			6	
3.67			6	
3.92				9
3.93				6
3.76				6
3.77				6
Total	73	73	70	70

left polar decompositions. For simplicity, we shall assume no change in elastic properties during plastic deformation and that the material is characterized by a bilinear Cauchy stress-engineering strain relation of the type shown in Fig. 3.5a.

Under these assumptions, we establish as possible configurations those shown in Fig. 3.5a. The reference, the current and the stress free configurations are shown and were computed using the definitions given earlier. An "elastic" configuration, which is merely another intermediate configuration, is introduced for convenience for calculating the Nemat-Nasser strain measures.

Here, we consider a specimen whose initial length is ℓ_o and, after a uniaxial homogeneous elastoplastic deformation due to a stress loading of σ, the specimen has elongated a length $\alpha\ell_o$ as indicated in Fig. 3.5b. Then an unloading causes the removal of the elastic portion of the resulting total deformation, i.e.,

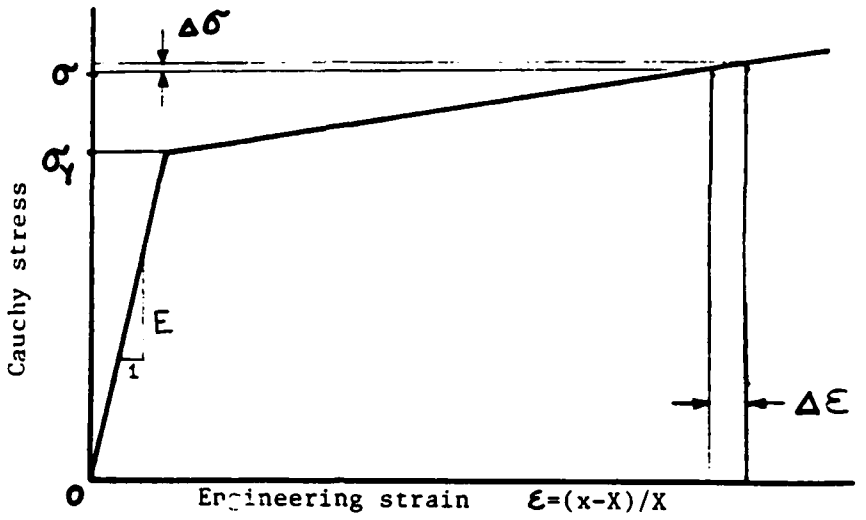
$$x = \alpha X \ , \quad 0 \leq X \leq \ell_o \tag{3.94}$$

$$\frac{\Delta e}{\alpha x - \Delta e} = \frac{\sigma}{E} \Rightarrow \Delta e = \frac{\alpha\sigma X}{E+\sigma} \tag{3.95}$$

Therefore, we have

$$P = \alpha X - \frac{\alpha\sigma X}{E+\sigma} = \frac{\alpha E X}{E+\sigma} \tag{3.96}$$

$$e = X + \frac{\alpha\sigma X}{E+\sigma} \tag{3.97}$$

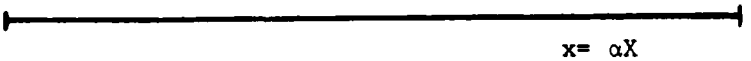


(a)

1) Reference configuration (C_o)

$0 \leq X \leq \ell_o$

2) Current configuration (C_t)



3) Stress free configuration (C_p)

4) Elastic configuration (C_e)

(b)

Figure 3.5 (a) Bilinear stress-strain relation.
(b) Configurations in one dimensional deformation.

Next, let us calculate $\Delta x, \Delta p$ and Δe due to a small change in σ, so that the increments (or rates) of various deformation gradients can be obtained:

$$x(\sigma + \Delta\sigma) = \alpha X + \frac{100\Delta\sigma X}{E} \equiv \bar{x}$$

$$P(\sigma + \Delta\sigma) = \alpha X + \frac{100\Delta\sigma X}{E} - \frac{(\sigma+\Delta\sigma)\bar{x}}{E+\sigma+\Delta\sigma} = \frac{\alpha E+100\Delta\sigma}{E+\sigma+\Delta\sigma} X$$

$$e(\sigma + \Delta\sigma) = X + \frac{(\sigma+\Delta\sigma)\bar{x}}{E+\sigma+\Delta\sigma}$$

Therefore

$$\Delta x = x(\sigma+\Delta\sigma) - x(\sigma) = \frac{100\Delta\sigma X}{E} \tag{3.98}$$

$$\Delta p = p(\sigma+\Delta\sigma) - p(\sigma)$$

$$= \frac{\alpha\sigma}{E+\sigma} X + \frac{100\Delta\sigma}{E} X - \frac{(\sigma+\Delta\sigma)\bar{x}}{E+\sigma+\Delta\sigma} \tag{3.99}$$

$$\Delta e = e(\sigma+\Delta\sigma) - e(\sigma)$$

$$= \frac{(\sigma+\Delta\sigma)\bar{x}}{E+\sigma+\Delta\sigma} - \frac{\alpha\sigma}{E+\sigma} X \tag{3.100}$$

We can now compare the measures of the two decomposition schemes, which are different in this one dimensional deformation.

i) Nemat - Nasser's Measures

$$F = \frac{\partial x}{\partial X} = \alpha \tag{3.101}$$

$$F^P = \frac{\partial P}{\partial X} = \alpha - \frac{\alpha\sigma}{E+\sigma} = \frac{\alpha E}{E+\sigma} \tag{3.102}$$

$$F^e = \frac{\partial e}{\partial X} = 1 + \frac{\alpha\sigma}{E+\sigma} \tag{3.103}$$

and

$$\Delta F^e = \dot{F}^e \Delta t = F^e(\sigma+\Delta\sigma) - F^e(\sigma)$$

$$= \frac{\partial \Delta e}{\partial X} = \frac{(\sigma+\Delta\sigma)}{E+\sigma+\Delta\sigma} \frac{\partial \bar{x}}{\partial X} - \frac{\alpha\sigma}{E+\sigma} \tag{3.104}$$

$$D^e \Delta t = \dot{F}^e \Delta t \; F^{-1}$$

$$= \Delta F^e \; F^{-1}$$

$$= \frac{(\sigma+\Delta\sigma)}{E+\sigma+\Delta\sigma} \left[1 + \frac{100\Delta\sigma}{\alpha E}\right] - \frac{\sigma}{E+\sigma}$$

$$= \frac{1}{(E+\sigma)(E+\sigma+\Delta\sigma)} \left(E\Delta\sigma + \frac{100\sigma\, E\Delta\sigma}{\alpha E} + \frac{100\sigma^2\Delta\sigma}{\alpha E}\right.$$

$$\left. + \frac{100E(\Delta\sigma)^2}{\alpha E} + \frac{100\sigma(\Delta\sigma)^2}{\alpha E}\right)$$

As $\Delta\sigma \to 0$ and $\Delta t \to 0$,

$$D^e = \frac{\dot{\sigma}}{(E+\sigma)^2} \left(E + \frac{100\sigma E}{\alpha E} + \frac{100\sigma^2}{\alpha E}\right) \tag{3.105}$$

$$= \frac{\sigma}{\alpha E(E+\sigma)^2} \left(\alpha E^2 + 100\, E\sigma + 100\sigma^2\right) \tag{3.106}$$

Similarly,

$$\Delta F^p = \frac{\partial \Delta P}{\partial x} = \frac{\alpha\sigma}{E+\sigma} + \frac{100\Delta\sigma}{E} - \frac{(\sigma+\Delta\sigma)}{E+\sigma+\Delta\sigma} \frac{\partial \bar{x}}{\partial X} \tag{3.107}$$

and

$$D^p = \frac{\sigma}{\alpha E(E+\sigma)^2} \left(100E^2 + 100E\sigma - \alpha E^2\right) \tag{3.108}$$

ii) Lee's Measures

$$\Delta e = e(\sigma+\Delta\sigma) - e(\sigma) = \frac{(\sigma+\Delta\sigma)\bar{x}}{E+\sigma+\Delta\sigma} - \frac{\alpha\sigma}{E+\sigma} X$$

$$F = \frac{\partial x}{\partial X} = \alpha$$

$$\bar{F}^e = \frac{E+\sigma}{E} \tag{3.109}$$

and

$$\Delta\bar{F}^e = \dot{\bar{F}}^e \Delta t = F^e(\sigma+\Delta\sigma) - F^e(\sigma)$$

$$= \frac{\partial x(\sigma+\Delta\sigma)}{\partial p(\sigma+\Delta\sigma)} - \bar{F}^e(\sigma)$$

$$= \frac{E+\sigma+\Delta\sigma}{E} - \frac{E+\sigma}{E} = \frac{\Delta\sigma}{E} \tag{3.110}$$

$$\bar{D}^e \Delta t = \dot{\bar{F}}^e \Delta t \bar{F}^{e-1}$$

$$= \frac{\Delta\sigma}{E+\sigma}$$

As $\Delta\sigma \to 0$ and $\Delta t \to 0$, then we have

$$D^e = \frac{\dot{\sigma}}{E+\sigma} \tag{3.111}$$

And for D^p (or ΔF^p) we have same result (3.107) [or (3.108)].

Since the plastic strain measures in the previous two cases are of same form, we now compare the elastic strain rates for various situations. Table 3.3 shows the comparisons for various values of α with fixed ratio of $\sigma/E = .001$ and Table 3.4 shows the comparisons

Table 3.3 Tabulation of proportional rates for $\sigma/E = .001$

α \ Proportional rate	$D^e E/\dot{\sigma}$	$\bar{D}^e E/\dot{\sigma}$
.2	1.4975	.999
.4	1.2477	.999
.8	1.1229	.999
1.0	1.0979	.999
1.5	1.0646	.999
2	1.0480	.999
3	1.0313	.999
4	1.0180	.999
5	1.0180	.999

Table 3.4 Tabulation of proportional rates for $x = 2$

σ/E \ Proportional rate	$D^e E/\dot{\sigma}$	$\bar{D}^e E/\dot{\sigma}$
10^{-6}	1.000048	.999999
10^{-5}	1.00048	.99999
10^{-4}	1.0048	.9999
10^{-3}	1.0479	.999
10^{-2}	1.475	.99
10^{-1}	5.37	.91
1	25.25	.5

for various values of σ/E with a fixed value of $\alpha=2$. In each Table, the values of D^eE/σ are tabulated. In other words, $D^eE/\sigma = 1$ means that

$$\dot{\sigma} = ED^e \text{ (or } E\bar{D}^e) \tag{3.112}$$

3.3.3 *Evaluation*

It appears to be difficult to define deformation, as shown in a convenient way that decompose into purely elastic and purely plastic parts even in one dimensional deformation as shown in Section 3.3.2. However, this is not actually a necessity in a meaningful theory, provided the constitutive equations for each are formulated properly (NEMAT-NASSER [1982], SIMO and ORTIZ [1985]). A consideration that may be of some importance is whether or not the particular decomposition lends itself to a well-determined system of equations and to rotation effect-free strain measures. This proves to be the case in the new formulations given here. A positive feature of Nemat-Nasser's decomposition, shared by our own right-polar decomposition, is that it does lend itself to experimental verification because of the normalization by total deformation gradient (or total stretch) in deformations (e.g., $\underset{\sim}{D}^e = \dot{\underset{\sim}{F}}^e\underset{\sim}{F}^{-1}|_{sym}$ or $\underset{\sim}{D}^e = \dot{\underset{\sim}{U}}^e\underset{\sim}{U}^{-1}|_{sym}$) which may be easily measured

in controlled laboratory experiments.

From now on, we employ the following strain measures:

$$\underset{\sim}{E} = \int_0^t \dot{\underset{\sim}{U}}^e \underset{\sim}{U}^{-1} \Big|_{sym} dt \tag{3.66}$$

$$\underset{\sim}{P} = \int_0^t \dot{\underset{\sim}{U}}^p \underset{\sim}{U}^{-1} \Big|_{sym} dt \tag{3.67}$$

CHAPTER IV

REPRESENTATIONS OF FLOW POTENTIALS AND APPLICATIONS

4.1 Introduction

In Chapter 3, a theory of elastoplasticity was presented that made use of the notion of generalized stress potentials for deriving constitutive equations for the rate-of-plastic deformation and the rate-of-change of an internal state variable. This theory embodies many types of classical and non-classical plasticity theories and does not require the existence of yield functions. Yield functions can enter the theory, however, through the definition of flow potentials, but these potentials need not be convex nor need they be differentiable in the usual sense.

Specific forms of the free energy and flow potentials can be easily derived for cases in which these are assumed to be isotropic functionals of their arguments. Such isotropic function representations are by no means unduly restrictive, for we demonstrate that they can be used to represent some anisotropic hardening rules. We show that numerous special theories of finite elastoplasticity and viscoplasticity that have been proposed in the literature can be deduced from such representations as special cases.

An interesting by-product of some of these flow potential representations arises because they do not specifically involve a yield function, but rather model the evolution of microstructural

changes in elastoplastic and viscoplastic materials during various programs of loading. As a result, by defining a yield stress as that corresponding to a 0.2% residual strain (in a uniaxial test) in much the same spirit as one defines it in an experimental program, we can actually calculate yield surfaces for such materials. Some of these mathematically produced yield surfaces are physically reasonable, others are not.

When the representation of flow potential happens to be determined by a given yield function, it is called an associative flow rule; otherwise, it is called a non-associative flow rule. Classical theories of plasticity are generally characterized by associative flow rule (DRUCKER 1967] and PRAGER [1959], etc.); while some authors (eg., ILYUSHIN [1960] and NEMAT-NASSER [1983]) have shown cases in which associative flow rules are inappropriate. ILYUSHIN also put forth arguments against the normality rule which arose from dropping the assumption that no change in elastic properties takes place during plastic deformation. NEMAT-NASSER introduced a notion of a "workless strain rate", which could invalidate the usual normality rule in calculating the rate of stress-work in the isothermal processes.

Another approach was advocated by HART [1970, 1976, 1982] and BODNER and coworkers [1972, 1975, 1979, 1983], who proposed plastic constitutive relations which involve a set of internal state variables, but not necessarily an explicit yield function.

Strain rate and rate-of-internal-state variables are independent constitutive variables in such theories.

A general representation of flow potentials for initially isotropics materials with internal state variables was proposed by KRATOCHVIL]1978] and KIM and ODEN [1985a].

4.2 Isotropic Functions

We begin by reviewing some notions of isotropic functions.

Definition 2.1: Let $Q \in \mathbb{R}^3 \times \mathbb{R}^3$ be an orthogonal transformation. A function $T : U \to V$, where U (or V) is one of the sets $\{\mathbb{R}, \mathbb{R}^3, \mathbb{R}^3 \times \mathbb{R}^3\}$, is said to be isotropic if

$$T^*(x) = T(x^*) \quad \forall \; x \in U \tag{4.1}$$

where $*$ is defined by

i) $B^* = B, \forall \; B \in \mathbb{R}$

ii) $\underset{\sim}{B}^* = \underset{\sim\sim}{Q}\underset{\sim}{B}$, $\forall \; \underset{\sim}{B} \in \mathbb{R}^3$

iii) $\underset{\sim}{B}^* = \underset{\sim\sim\sim}{Q A Q^T}$, $\forall \; \underset{\sim}{B} \in \mathbb{R}^3 \times \mathbb{R}^3$

Next we list the representation theorems for isotropic functions without proof. (For proofs, see GURTIN [1982], LEIGH [1968] or TRUESDELL and NOLL [1965]).

Let G be a subgroup of the symmetric second-order tensor group and S_A be the set of invariants of $\underset{\sim}{A}$, i.e.,

$$\underset{\sim}{I}_A = \{\text{tr } \underset{\sim}{A}, \; \text{tr } \underset{\sim}{A}^2, \; \text{tr } \underset{\sim}{A}^3\}$$

Let $I(G)$ be the set

$$I(G) = \{I_{\underset{\sim}{A}},\ \underset{\sim}{A} \in G\}$$

Similarly, we denote

$$I_{AB} = \{\text{tr } \underset{\sim}{A},\ \text{tr } \underset{\sim}{B},\ \text{tr } \underset{\sim}{A}^2,\ \text{tr } \underset{\sim}{B}^2,\ \text{tr } \underset{\sim}{A}^3,\ \text{tr } \underset{\sim}{B}^3\ \text{tr } \underset{\sim}{AB},$$
$$\text{tr } \underset{\sim}{A}\,\underset{\sim}{B}^2,\ \text{tr } \underset{\sim}{A}^2\underset{\sim}{B},\ \text{tr } \underset{\sim}{A}^2\underset{\sim}{B}^2\}$$

and

$$I(G \times G) = \{I_{AB},\ A \times B \in G \times G\}$$

(1) Representation Theorem for Isotropic Scalar Functions

a) A function $\phi : G \to \mathbb{R}$ is isotropic if there exists a function $\tilde{\phi} : I(G) \to \mathbb{R}$ such that

$$\phi(\underset{\sim}{A}) = \tilde{\phi}(I_{\underset{\sim}{A}}),\ \forall\ \underset{\sim}{A} \in G \tag{4.2}$$

b) A function $\phi : G \times G \to \mathbb{R}$ is isotropic if there exists a function $\tilde{\phi}: I(G \times G) \to \mathbb{R}$ such that

$$\phi(\underset{\sim}{A},\ \underset{\sim}{B}) = \tilde{\phi}(I_{AB}),\ \forall \underset{\sim}{A} \times \underset{\sim}{B} \in G \times G \tag{4.3}$$

(2) Representation Theorems for Isotropic Tensor Functions

a) A tensor-valued function $\underset{\sim}{G} : G \to S$ is isotropic if there exist scalar functions $\phi_0,\ \phi_1\ \phi_2 ; I(G) \to \mathbb{R}$ such that

$$\underset{\sim}{G}(\underset{\sim}{A}) = \phi_0(I_{\underset{\sim}{A}})\ \underset{\sim}{1} + \phi_1(I_{\underset{\sim}{A}})\underset{\sim}{A} = \phi_2(I_{\underset{\sim}{A}})\underset{\sim}{A}^2,\ \forall\ \underset{\sim}{A} \in G \tag{4.4}$$

b) For isotropic linear tensor functions, the function $\underset{\sim}{G} : S \to S$ is isotropic if there exist scalars μ and λ such that

$$\underset{\sim}{G}(\underset{\sim}{A}) = 2\mu\underset{\sim}{A} + \lambda(\text{tr } \underset{\sim}{A})\underset{\sim}{1}\ ,\ \forall\ \underset{\sim}{A} \in S \tag{4.5}$$

c) A tensor-valued function $\underset{\sim}{G} : G \times G \to S$ is isotropic if there exist scalar functions ϕ_i, $i = 1,2,\ldots,8$, $: I(G \times G) \to \mathbb{R}$ such that

$$\underset{\sim}{G}(\underset{\sim}{A}, \underset{\sim}{B}) = \phi_0 \underset{\sim}{1} + \phi_1 \underset{\sim}{A} + \phi_2 \underset{\sim}{B} + \phi_3 \underset{\sim}{A}^2 + \phi_4 \underset{\sim}{B}^2 + \phi_5 (\underset{\sim}{A}\underset{\sim}{B} + \underset{\sim}{B}\underset{\sim}{A})$$

$$+ \phi_6 (\underset{\sim}{A}^2\underset{\sim}{B} + \underset{\sim}{B}\underset{\sim}{A}^2) + \phi_7 (\underset{\sim}{A}\underset{\sim}{B}^2 + \underset{\sim}{B}^2\underset{\sim}{A}) + \phi_8 (\underset{\sim}{A}^2\underset{\sim}{B}^2 + \underset{\sim}{B}^2\underset{\sim}{A}^2)$$

$$\forall (\underset{\sim}{A}, \underset{\sim}{B}) \in G \times G \qquad (4.6)$$

It is worthwhile to mention the definition of an isotropic material as opposed to an isotropic function to perceive relevancy.

An isotropic material particle is one such that for some reference configuration C_0, all rotations by the response functions of the material are indiscernable (cf. WANG and TRUESDELL [1973]). In other words, the material is isotropic at particle p if every proper orthogonal tensor is a symmetry transformation for the response functions of the material (GURTIN [1981b], or for a slightly different version see ERINGEN [1962, 1967]).

4.3 Representation of Flow Potential and Free Energy

From the theory of materials of type N, we obtained the plastic strain-rate and the-rate-of-internal state variable, through relations (3.45) and (3.48) respectively in the differential case, from the flow potential. Next questions that arise are how one might obtain a specific flow potential for a given material and what might be considered

as general forms of flow potentials for certain classes of materials.

Towards this end, we shall first confine our attention to materials characterized by isotropic functionals. The following approach is similar to that of KRATOCHVIL [1978], although he did not make use of the concept of flow potentials. We also mention that DESAI et al. [1983] suggested the use of polynominal expansions of flow potentials in terms of invariants of its arguments.

From the equation (4.3), an isotropic representation of $\psi(\underset{\sim}{\sigma}, \underset{\sim}{A})$ is of the form,

$$\psi(\underset{\sim}{\sigma}, \underset{\sim}{A}) = \tilde{\psi}(\text{tr}\,\underset{\sim}{\sigma}, \text{tr}\,\underset{\sim}{A}, \text{tr}\,\underset{\sim}{\sigma}^2, \text{tr}\,\underset{\sim}{A}^2, \text{tr}\,\underset{\sim}{\sigma}^3, \text{tr}\,\underset{\sim}{A}^3, \text{tr}\,\underset{\sim}{\sigma}\underset{\sim}{A}, \text{tr}\,\underset{\sim}{\sigma}\underset{\sim}{A}^2, \text{tr}\,\underset{\sim}{\sigma}^2\underset{\sim}{A}, \text{tr}\,\underset{\sim}{\sigma}^2\underset{\sim}{A}) \tag{4.7}$$

It is occasionally convenient in classical plasticity theories to decompose the tensors $\underset{\sim}{\sigma}$, $\underset{\sim}{A}$ into spherical and deviatoric parts, i.e.,

$$\begin{aligned} \underset{\sim}{\sigma} &= s\underset{\sim}{1} + \overline{\underset{\sim}{S}} \\ \underset{\sim}{A} &= h\underset{\sim}{1} + \underset{\sim}{H} \\ \underset{\sim}{P} &= p\underset{\sim}{1} + \overline{\underset{\sim}{P}} \\ \underset{\sim}{\alpha} &= \lambda\underset{\sim}{1} + \underset{\sim}{\Lambda} \end{aligned} \tag{4.8}$$

where $s = \text{tr}\,\underset{\sim}{\sigma}/3$, $h = \text{tr}\,\underset{\sim}{A}/3$, $p = \text{tr}\,\underset{\sim}{P}/3$ and $\lambda = \text{tr}\,\underset{\sim}{\alpha}/3$. Then, the expression (4.7) can be written by

$$\psi(\underset{\sim}{\sigma}, \underset{\sim}{A}) = \bar{\psi}(s,\ h,\ \text{tr}\ \bar{\underset{\sim}{S}}^2,\ \text{tr}\ \underset{\sim}{H}^2,\ \text{tr}\ \bar{\underset{\sim}{S}}^3,\ \text{tr}\ \underset{\sim}{H}^3,\ \text{tr}\ \bar{\underset{\sim}{S}}\underset{\sim}{H},$$

$$\text{tr}\ \bar{\underset{\sim}{S}}\ \underset{\sim}{H}^2,\ \text{tr}\ \bar{\underset{\sim}{S}}^2\underset{\sim}{H}^2,\ \text{tr}\ \bar{\underset{\sim}{S}}^2\underset{\sim}{H}^2) \tag{4.9}$$

The following lemma can be easily proved.

LEMMA 4.1: Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be second order symmetric tensors and be decomposed according to

$$a = \text{tr}\ \underset{\sim}{A}/3\ ,\quad \bar{\underset{\sim}{A}} = \underset{\sim}{A} - a\underset{\sim}{1}$$

$$b = \text{tr}\ \underset{\sim}{B}/3\ ,\quad \bar{\underset{\sim}{B}} = \underset{\sim}{B} - b\underset{\sim}{1}$$

then

$$\frac{\partial a}{\partial \underset{\sim}{A}} = \frac{1}{3}\underset{\sim}{1}\ ,\quad \frac{\partial(\text{tr}\ \bar{\underset{\sim}{A}}^2)}{d\underset{\sim}{A}} = 2\bar{\underset{\sim}{A}}$$

$$\frac{\partial(\text{tr}\ \bar{\underset{\sim}{A}}\bar{\underset{\sim}{B}})}{\partial \underset{\sim}{A}} = \bar{\underset{\sim}{B}} = \frac{\partial(\text{tr}\ \bar{\underset{\sim}{B}}\bar{\underset{\sim}{A}})}{\partial \underset{\sim}{A}}$$

$$\frac{\partial(\text{tr}\ \bar{\underset{\sim}{A}}^2\bar{\underset{\sim}{B}})}{\partial \underset{\sim}{A}} = (\bar{\underset{\sim}{A}}\bar{\underset{\sim}{B}}) + (\bar{\underset{\sim}{B}}\bar{\underset{\sim}{A}}) - \frac{2}{3}(\text{tr}\ \bar{\underset{\sim}{A}}\bar{\underset{\sim}{B}})\ \underset{\sim}{1} = \frac{\partial \text{tr}(\bar{\underset{\sim}{A}}\bar{\underset{\sim}{B}}^2)}{\partial \underset{\sim}{B}}$$

$$\frac{\partial \text{tr}(\bar{\underset{\sim}{A}}^3)}{\partial \underset{\sim}{A}} = 3\bar{\underset{\sim}{A}}^2 - \text{tr}(\bar{\underset{\sim}{A}}^2)\ \underset{\sim}{1}$$

$$\frac{\partial \text{tr}(\bar{\underset{\sim}{A}}^2\bar{\underset{\sim}{B}})}{\partial \underset{\sim}{B}} = \bar{\underset{\sim}{A}}^2 - \frac{1}{3}\text{tr}\ (\bar{\underset{\sim}{A}}^2)\ \underset{\sim}{1} = \frac{\partial \text{tr}(\bar{\underset{\sim}{B}}\bar{\underset{\sim}{A}}^2)}{\partial \underset{\sim}{B}}$$

$$\frac{\partial \text{tr}(\bar{\underset{\sim}{A}}^2\bar{\underset{\sim}{B}}^2)}{\partial \underset{\sim}{A}} = (\bar{\underset{\sim}{A}}\ \bar{\underset{\sim}{B}}^2) + (\bar{\underset{\sim}{B}}^2\ \bar{\underset{\sim}{A}}) - \frac{2}{3}\text{tr}(\bar{\underset{\sim}{A}}\ \bar{\underset{\sim}{B}}^2)\underset{\sim}{1} = \frac{\partial \text{tr}(\bar{\underset{\sim}{B}}^2\bar{\underset{\sim}{A}}^2)}{\underset{\sim}{A}}$$

When $\psi(\underset{\sim}{\sigma}, \underset{\sim}{A})$ is differentiable with respect to both arguments at $(\underset{\sim}{\sigma}, \underset{\sim}{A})$, by using equation and Lemma 4.1, we can establish the following theorem.

THEOREM 4.1: If the representation of $\psi(\underset{\sim}{\sigma}, \underset{\sim}{A})$ is of the form of equation (4.9) and $\psi : \bar{w} \to \mathbb{R}$ is differentiable at $(\underset{\sim}{\sigma}, \underset{\sim}{A})$, then

$$\dot{\underset{\sim}{P}} = C_0\underset{\sim}{1} + C_1\underset{\sim}{\bar{S}} + C_2\underset{\sim}{H} + C_3\underset{\sim}{\bar{S}}^2 + C_4\underset{\sim}{H}^2 + C_5(\underset{\sim}{H}\underset{\sim}{\bar{S}} + \underset{\sim}{\bar{S}}\underset{\sim}{H})$$

$$+C_6(\underset{\sim}{\bar{S}}\ \underset{\sim}{H}^2 + \underset{\sim}{H}^2\underset{\sim}{\bar{S}}) \tag{4.10}$$

and

$$\dot{\underset{\sim}{\alpha}} = d_0\underset{\sim}{1} + d_1\underset{\sim}{H} + d_2\underset{\sim}{\bar{S}} + d_3\underset{\sim}{H}^2 + d_4\underset{\sim}{\bar{S}}^2 + d_5(\underset{\sim}{H}\underset{\sim}{\bar{S}} + \underset{\sim}{\bar{S}}\underset{\sim}{H})$$

$$+d_6(\underset{\sim}{H}\underset{\sim}{\bar{S}}^2 + \underset{\sim}{\bar{S}}^2\underset{\sim}{H}) \tag{4.11}$$

where C_i's and d_i's are scalar valued functions of the invariants, i.e., $I_{\sigma A}$.

Proof: The chain rule, Lemma 3.1 and symmetries of $\underset{\sim}{S}$ and $\underset{\sim}{H}$ suffice to prove the theorem. □

A similar representation of the free energy ψ of the material can also be introduced; e.g.,

$$\phi(\underset{\sim}{E}, \underset{\sim}{\alpha}) = \bar{\varphi}(e, \underset{\sim}{\Lambda}\ \text{tr}\ \underset{\sim}{\bar{E}}^2, \text{tr}\ \underset{\sim}{\Lambda}^2, \text{tr}\ \underset{\sim}{\bar{E}}^3, \text{tr}\ \underset{\sim}{\Lambda}^3, \text{tr}\ \underset{\sim}{\bar{E}}\ \underset{\sim}{\Lambda},$$

$$\text{tr}\ \underset{\sim}{\bar{E}}\underset{\sim}{\Lambda}^2, \text{tr}\ \underset{\sim}{\bar{E}}^2\underset{\sim}{\Lambda}, \text{tr}\ \underset{\sim}{\bar{E}}^2\underset{\sim}{\Lambda}^2) \tag{4.12}$$

where $\underset{\sim}{E}$ is given as a sum

$$\underset{\sim}{E} = e\underset{\sim}{1} + \underset{\sim}{\bar{E}} \text{ with } e = \frac{1}{3}\ \text{tr}\ \underset{\sim}{E}$$

From the relations (3.61) and (3.59) [or (3.64) or (3.70) or (3.78)], we have

$$\underset{\sim}{A} = k_0\underset{\sim}{1} + k_1\underset{\sim}{\Lambda} = k_2\underset{\sim}{\bar{E}} + k_3\underset{\sim}{\Lambda}^2 + k_4\underset{\sim}{\bar{E}}^2 + k_5(\underset{\sim}{\bar{E}}\underset{\sim}{\Lambda} + \underset{\sim}{\Lambda}\underset{\sim}{\bar{E}}) + k_6(\underset{\sim}{\bar{E}}^2\underset{\sim}{\Lambda} + \underset{\sim}{\Lambda}\underset{\sim}{\bar{E}}^2) \tag{4.13}$$

$$\underset{\sim}{\sigma} = \ell_0\underset{\sim}{1} + \ell_1\underset{\sim}{\bar{E}} + \ell_2\underset{\sim}{\Lambda} + \ell_3\underset{\sim}{\bar{E}}^2 + \ell_4\underset{\sim}{\Lambda}^2 + \ell_5(\underset{\sim}{\Lambda}\underset{\sim}{\bar{E}} + \underset{\sim}{\bar{E}}\underset{\sim}{\Lambda}) + \ell_6(\underset{\sim}{\Lambda}^2\underset{\sim}{\bar{E}} + \underset{\sim}{\bar{E}}\underset{\sim}{\Lambda}^2) \tag{4.14}$$

As before, the k_i's and ℓ_i's are scalar functions of the invariants and density ρ. We also note that, in many cases, it is convenient to introduce a decomposition of the free energy into purely elastic and plastic parts, i.e.,

$$\phi(\underset{\sim}{E}, \underset{\sim}{\alpha}) = \phi^e(\underset{\sim}{E}) + \phi^p(\underset{\sim}{\alpha}) \tag{4.15}$$

If (4.15) holds, equations (4.13) and (4.14) reduce to

$$\underset{\sim}{A} = k_0\underset{\sim}{1} + k_1\underset{\sim}{\Lambda} + k_3\underset{\sim}{\Lambda}^2 \tag{4.16}$$

$$\underset{\sim}{\sigma} = \ell_0\underset{\sim}{1} + \ell_1\underset{\sim}{\bar{E}} + \ell_3\underset{\sim}{\bar{E}}^2 \tag{4.17}$$

REMARK 4.1: The inclusion of temperature does not result in a significant alteration in our procedure. For example, if we rewrite the equation (4.15) (as in KRONER and TEODOSIU [1974]) in the form,

$$\phi(\underset{\sim}{E}, \underset{\sim}{\alpha}, T) = \phi^e(\underset{\sim}{E}, T) + \phi^p(\underset{\sim}{\alpha}, T) \tag{R1}$$

we would obtain same form as (4.13) and (4.14). But the coefficients become functions of temperature. □

REMARK 4.2: One of the representations of ϕ may be drawn from linear elasticity, i.e.,

$$\phi^e = \frac{1}{2}\lambda\,(\mathrm{tr}\,\underset{\sim}{E})^2 + \mu\,\mathrm{tr}\,(\underset{\sim}{E})^2 \tag{R2}$$

where λ and μ are Lame's constants. The elastic free energy (R2) results in the relation

$$\underset{\sim}{\sigma} = \lambda\,(\mathrm{tr}\,\underset{\sim}{E})\,\underset{\sim}{1} + 2\mu\,\underset{\sim}{E} \tag{R3}$$

Since (R3) is invertible, we obtain

$$\underset{\sim}{E} = \frac{1+\nu}{E}\,\underset{\sim}{\sigma} - \frac{\nu}{E}\,\mathrm{tr}\,\underset{\sim}{\sigma}\,\underset{\sim}{1} \tag{R4}$$

which results in

$$\dot{\underset{\sim}{E}} = \frac{1+\nu}{E}\,\overset{\nabla}{\underset{\sim}{\sigma}} - \frac{\nu}{E}\,\mathrm{tr}\,\overset{\nabla}{\underset{\sim}{\sigma}}\,\underset{\sim}{1} \tag{R5}$$

where E and ν are Young's modulus and Poisson ratio, respectively, and $\overset{\nabla}{\underset{\sim}{\sigma}}$ is an appropriate objective stress rate of Cauchy stress $\underset{\sim}{\sigma}$, and $\underset{\sim}{E}$ is the elastic strain measure of (3.66). □

REMARK 4.3: It is noted that in the present theoretical framework, two representations of scalar functions, the flow potential and free energy, are used as compared to only one, free energy, in some thermodynamic approaches (e.g., KRATOCHVIL and DILLON [1969]). But these approaches are equivalent to ours because one must ultimately provide the evolution equations for the internal state variables and the plastic strain rate. These are to be determined by the flow potential ψ in our theory. □

4.4 Examples

We now investigate how various theories of plasticity fit into the above theory.

4.4.1 *Case I: Prandtl-Reuss Flow Rule*

The following representations recover the Prandtl-Reuss flow rule which is also obtained by defining the Levy-Mises yield function as a flow potential. Assuming (4.15) holds, set

$$\psi = \psi(h, \operatorname{tr} \bar{\underset{\sim}{S}}^2) \qquad \phi^P = \phi^P(\lambda) \tag{4.18}$$

When the functionals in (4.18) are differentiable with their arguments, we have

$$\dot{\underset{\sim}{P}} = C_1 \bar{\underset{\sim}{S}} \tag{4.19}$$

$$\dot{\underset{\sim}{\alpha}} = C_0 \underset{\sim}{1} \Rightarrow \dot{\lambda} = C_0 \tag{4.20}$$

and

$$\underset{\sim}{A} = k_0 \underset{\sim}{1} \Rightarrow h = k_0 \tag{4.21}$$

Set $C_1 = \dot{\lambda} = C_0$. Then (4.19) reduces to

$$\dot{\underset{\sim}{P}} = \dot{\lambda} \bar{\underset{\sim}{S}} \tag{4.22}$$

which is precisely the flow rule for the Levy-Mises material. In this case, h can be interpreted as the bound in the Levy-Mises criteria, i.e.,

$$f(\underset{\sim}{\sigma}) = \operatorname{tr} \bar{\underset{\sim}{S}}^2 \leq \sigma_Y \equiv h = k_0$$

When $h = k_0(\alpha)$, the representations (4.18) describes a rate-dependent plastic material with isotropic hardening.

4.4.2 *Case 2: Bell's Flow Rule*

If we set $\dot{\lambda} = 2 \operatorname{tr} \bar{\underset{\sim}{S}}^2/\beta^2$, where β is constant, then we recover Bell's flow rule for finite strain plasticity (see BELL [1983, 1984]).

4.4.3 *Case 3: The Prager-Ziegler Type Kinematic Hardening*

The representation

$$\left.\begin{aligned} \psi &= \psi(h, \operatorname{tr} \bar{\underset{\sim}{S}}^2, \operatorname{tr} \underset{\sim}{H}, \operatorname{tr} \bar{\underset{\sim}{S}}\, \underset{\sim}{H}) \\ \phi^P &= \phi^P(\lambda, \operatorname{tr} \underset{\sim}{\Lambda}^2) \end{aligned}\right\} \tag{4.23}$$

when ψ and ϕ^P are differentiable, leads to the following:

$$\dot{\underset{\sim}{P}} = C_1 \bar{\underset{\sim}{S}} + C_2 \underset{\sim}{H} \tag{4.24}$$

$$\dot{\underset{\sim}{\alpha}} = d_0 \underset{\sim}{1} + d_1 \underset{\sim}{H} + d_2 \bar{\underset{\sim}{S}} \tag{4.25}$$

and

$$\underset{\sim}{A} = k_0 \underset{\sim}{1} + k_1 \underset{\sim}{\Lambda} \tag{4.26}$$

Then equation (4.26) implies that

$$h = k_0, \quad \underset{\sim}{H} = k_1 \Lambda \tag{4.27}$$

Introducing (4.27) into (4.24) and (4.25) gives

$$\dot{\underset{\sim}{P}} = C_1 \bar{\underset{\sim}{S}} + f_1 \underset{\sim}{\Lambda} \tag{4.28}$$

and

$$\dot{\lambda} = d_0 \tag{4.29}$$

$$\dot{\underset{\sim}{\Lambda}} = f_2 \underset{\sim}{\Lambda} + d_2 \bar{\underset{\sim}{S}} \tag{4.30}$$

where $f_1 = C_2 k_1$ and $f_2 = d_1 k_1$.

We now demonstrate how Prager-Ziegler-type kinematic hardening is characterized by the representation (4.23). The Prager-Ziegler yield function may be written in the form

$$F(\bar{\underset{\sim}{S}}, \underset{\sim}{S}_b, \sigma_\gamma) \equiv \mathrm{tr}(\bar{\underset{\sim}{S}} - \underset{\sim}{S}_b)^2 - \sigma_\gamma^2 \tag{4.31}$$

where $\underset{\sim}{S}_b$ is the deviatoric back stress which represents the current center of the yield surface. The yield function (4.31) results in the flow rule,

$$\dot{\underset{\sim}{P}} = \dot{n}(\bar{\underset{\sim}{S}} - \underset{\sim}{S}_b) \tag{4.32}$$

where n is a hardening parameter. Next we set $C_1 = -f_1 = d_0 = \dot{\lambda} = \dot{n}$ and $\underset{\sim}{S}_b = \underset{\sim}{\Lambda}$. Then, equations (4.28), (4.29) and (4.30) give a set of equations which define a kinematic flow rule and an evolution equation for back stress or an internal state variable, respectively:

$$\dot{\underset{\sim}{P}} = \dot{\lambda}(\bar{\underset{\sim}{S}} - \underset{\sim}{\Lambda}) \tag{4.33}$$

and

$$\dot{\Lambda} = f_2 \Lambda + d_2 \bar{\underset{\sim}{S}} \tag{4.34}$$

REMARK 4.4: As noted in equations (4.24) - (4.26) or (4.28) - (4.30), the tensor valued functions $\underset{\sim}{D}^P$, $\dot{\alpha}$ and $\underset{\sim}{A}$ are isotropic even though they represent an anisotropic hardening material (see equation (4.6) with appropriate zero coefficients). This interesting fact, that

a non-isotropic hardening rule results from an isotropic representation of a flow potential functional or an isotropic form of the constitutive equations for $\underset{\sim}{D}^P$ and $\dot{\underset{\sim}{\Lambda}}$, suggests that references to anisotropic hardening may not be proper terminology. □

4.4.4 *Case 4: Flow Rule for Tresca Materials*

The Tresca yield criteria,

$$\max_{i,j} \mid \sigma_i - \sigma_j \mid - \sigma_Y \leq 0 \tag{4.35}$$

may be represented by the non-differentiable functionals

$$\psi = \psi(h, \text{ tr } \underset{\sim}{\bar{S}}^2, \text{ tr } \underset{\sim}{\bar{S}}^2) \tag{4.36}$$

$$\phi^P = \phi^P(\lambda)$$

The principal stresses σ_i are functions of tr $\underset{\sim}{\bar{S}}^2$ and tr $\underset{\sim}{\bar{S}}^3$ which satisfy the characteristic equation

$$\sigma_i^3 - \frac{1}{2} \text{tr}(\underset{\sim}{\bar{S}}^2)\sigma_i - \frac{1}{3} \text{tr}(\underset{\sim}{\bar{S}}^3) = 0 \tag{4.37}$$

Since the Tresca yield function is convex but non-differentiable at some stress points (the corners of the yield surfaces), the classical flow rule is characterized by

$$(\dot{\underset{\sim}{P}}, -\dot{\underset{\sim}{\alpha}}) \in \partial\psi(\underset{\sim}{\sigma}, \underset{\sim}{A}) \tag{4.38}$$

$$\underset{\sim}{A} = k_0 \underset{\sim}{1} \tag{4.39}$$

where $\partial\psi(\cdot,\cdot)$ is the subdifferential.

$$\partial\psi(\underset{\sim}{\sigma}, \underset{\sim}{A}) = \{(\dot{\underset{\sim}{P}}, -\dot{\underset{\sim}{\alpha}}) \mid \text{ satisfies the inequality (3.46)}\}.$$

One can imagine generalizations of the Tresca flow potential to cases in which ψ is neither convex nor differentiable. Consider a hypothetical material, the flow rule for which is characterized by a convex Tresca yield surface at a given point in its stress history. Suppose that low latent hardening prevails. Then subsequent anisotropic hardening, after loadings in two distinct direction as in Fig. 4.1, could conceivably produce a non-convex yield surface of the type shown in Fig. 4.1. While some may argue that such hardening patterns are rare or non-existent in real materials, they are nevertheless acceptable in the general theory outlined here. As an example of non-convex yield surface, WILKINS et al [1980] derived a star shaped yield surface such as that in Fig. 4.2 from experiments on metal specimens. □

REMARK 4.5: We also remark that there is frequently some ambiguity in the literature as to exactly what is meant by the yield surface of a given material. BELL [1983, 1984] uses two yield surfaces, the inner surface and outer surface, and HILL [1979] also has distinguished between surfaces corresponding to an elastic limit and a plastic limit. In subsequent examples, we use a single yield limit (or surface), as the elastic limit, for simplicity. □

Now let us investigate a proposed new model of plasticity (or viscoplasticity) which need not employ the concept of a yield surface.

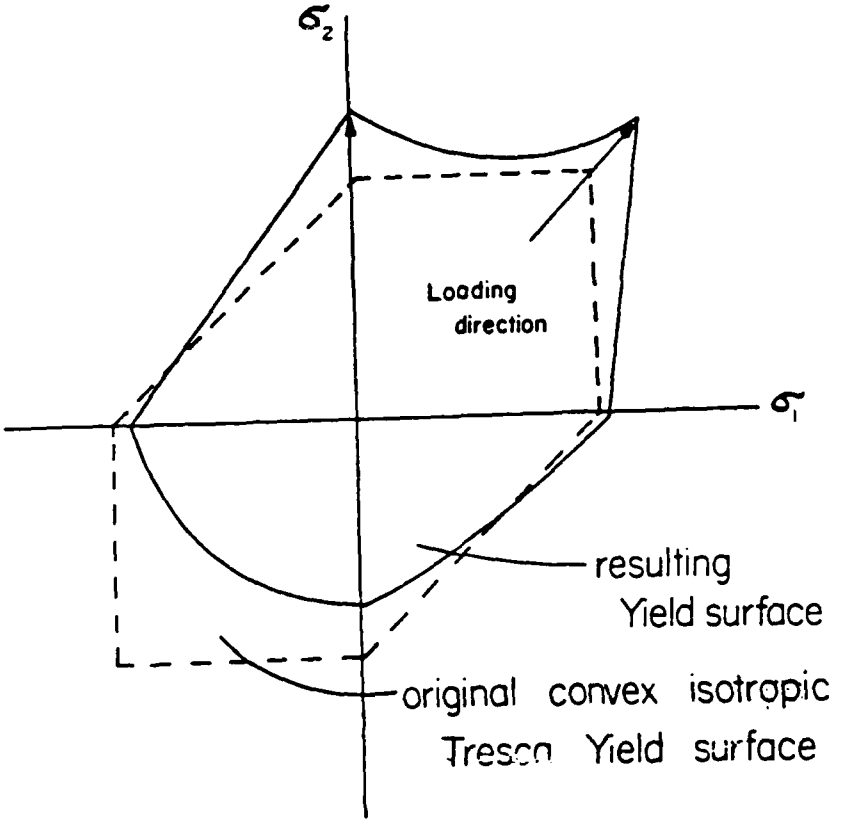


Figure 4.1 Subsequent yield surface after two loadings in different directions.

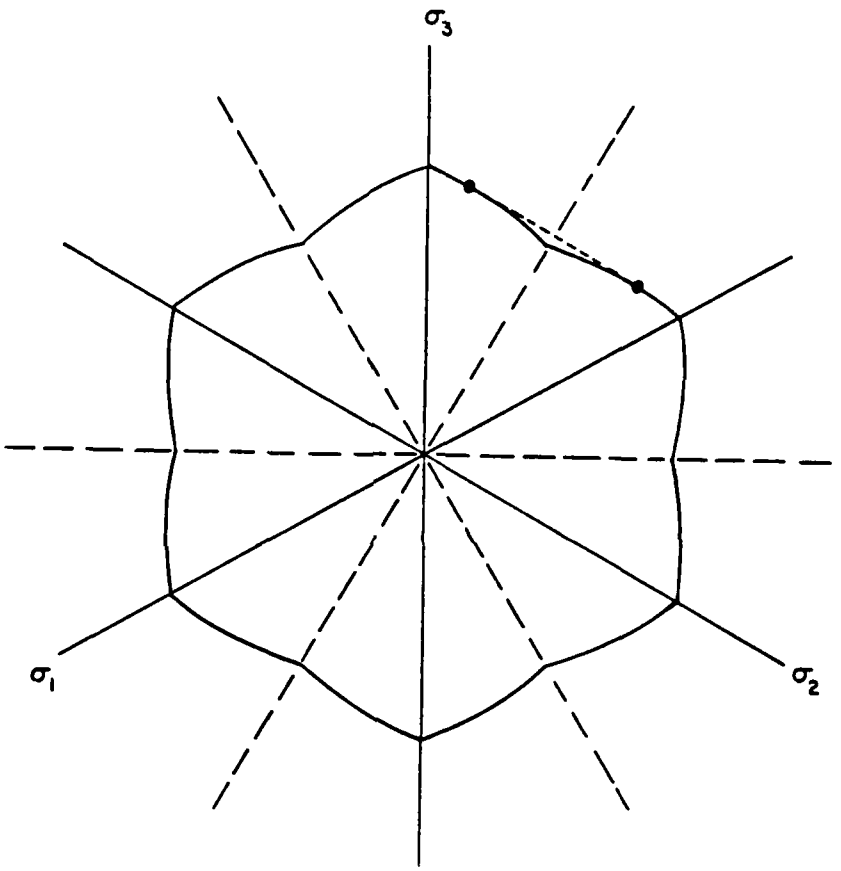


Figure 4.2 The nonconvex yield surface of WILKINS et al. [1980].

4.4.5 *Case 5: Bodner et al. Flow Rule*

4.4.5.1 *Isotropic Hardening*

BODNER and PARTOM [1972, 1975] proposed a flow rule which is drawn from the classical Prandtl-Reuss flow rule and improved by phenomenological observations, but which does not involve specification of a yield surface. This theory is offered as a way for characterizing isotropic hardening of certain materials.

For the Bodner/Partom theory, the following form of potentials can be introduced:

$$\psi = D_0^2 \sum_{i=0}^{\infty} \frac{(-1)^{i+1} B^{2ni}}{(2ni-1)i!\ J_2^{(ni-\frac{1}{2})}} \tag{4.40}$$

$$\phi^P = -h_1\lambda - h_0(h_1-h_0)\ \exp\ (-m\lambda/h_0)/m \tag{4.41}$$

Here

$$J_2 = \frac{1}{2}\ \mathrm{tr}\ \underset{\sim}{\bar{S}}^2$$

$$B^2 = \frac{1}{3}\ h^2\ \left(\frac{n+1}{n}\right)^{1/n}$$

and D_0, h_0, h_1 and n are material constants.

From the expressions (4.40), (4.41) and the generalized normality rule (3.45) (in this case, equation (3.48)), we obtain the following Bodner/Partom flow rule,

$$\left.\begin{aligned} \dot{\underset{\sim}{P}} &= (D_0^2 \exp[-B^2/J_2)^n])\underset{\sim}{S}/J_2^{\frac{1}{2}} \\ \text{with} \\ h &= h_1 + (h_0 - h_1)\exp(-m\lambda/h_0) \end{aligned}\right\} \tag{4.42}$$

It is noted that as $n \to \infty$, equation (4.42) models non-work hardening materials (i.e., the elastic perfect plastic material).

4.4.5.2 *Non-Isotropic Hardening*

STOUFFER and BODNER [1979] extended the previous hardening model to the case of non-isotropic materials. A plasticity theory for non-isotropic hardening can be constructed by introducing implicit forms of potentials which are different from the previous isotropic representations, i.e.,

$$\begin{aligned} \psi &= \psi(\underset{\sim}{\sigma}, \underset{\sim}{A}, \underset{\sim}{\alpha}_2) \\ \sigma^P &= \sigma(\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2) \end{aligned} \tag{4.43}$$

where $\psi \in C^1(\mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{R}^9)$ and $\sigma \in C^1(\mathbb{R}^9 \times \mathbb{R}^9)$, and we employ two internal state variables, $\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$. As one example, suppose that $\underset{\sim}{D}^P$ is given by a generalized Prandtl-Reuss flow rule

$$\left.\begin{aligned} \dot{\underset{\sim}{P}} &= \frac{\partial \psi}{\partial \underset{\sim}{\sigma}} = \underset{\sim}{K}(\underset{\sim}{A}_1) \\ \text{or} \\ \dot{P}_{ij} &= K_{ijk\ell}\,\sigma_{k\ell} \end{aligned}\right\} \tag{4.44}$$

$$\left.\begin{aligned} \underset{\sim}{A}_1 &= \frac{\partial \phi}{\partial \underset{\sim}{\alpha}_1} = P_0 \underset{\sim}{1} \Rightarrow h = P_0(\lambda_1) \\ A_2 &= \frac{\partial \phi}{\partial \alpha_2} = P_1(h)\underset{\sim}{1} + P_2(h)\underset{\sim}{\alpha}_2 \end{aligned}\right\} \tag{4.45}$$

The symmetry of $\underset{\sim}{D}^P$ and $\underset{\sim}{\sigma}$ and the reciprocity condition of HILL [1979] result in the conditions

$$K_{ijk\ell} = K_{jik\ell} = K_{ij\ell k} = K_{k\ell ijj} \tag{4.46}$$

If, in addition, we assume that the hydrostatic component of the stress does no work, then equation (4.44) becomes

$$\dot{\underset{\sim}{P}} = \underset{\sim}{K}(\underset{\sim}{A}_1)\bar{\underset{\sim}{S}} \text{ or } D^P_{ij} = K_{ijk}\ \bar{S}_k \tag{4.47}$$

Due to conditions (61), equation (62) can be rewritten in the form

$$\hat{\underset{\sim}{d}}^P = \hat{\underset{\sim}{\beta}}\ \hat{\underset{\sim}{T}} \tag{4.48}$$

where

$$\{d^P_i\} = \begin{Bmatrix} \dot{P}_{11} \\ \dot{P}_{22} \\ \dot{P}_{33} \\ \sqrt{2}\ \dot{P}_{12} \\ \sqrt{2}\ \dot{P}_{22} \\ \sqrt{2}\ \dot{P}_{33} \end{Bmatrix} \quad ,\{\hat{T}_i\} = \begin{Bmatrix} \bar{S}_{11} \\ \bar{S}_{22} \\ \bar{S}_{33} \\ \sqrt{2}\ \bar{S}_{12} \\ \sqrt{2}\ \bar{S}_{23} \\ \sqrt{2}\ \bar{S}_{31} \end{Bmatrix} \quad \text{and } \hat{\underset{\sim}{\beta}}^T = \hat{\underset{\sim}{\beta}}$$

Next, a principal hardness direction can be defined such that

$$d^P_\alpha = \beta_\alpha\ T_\alpha \quad (\alpha \text{ is not summed}) \tag{4.49}$$

so that we can rewrite equation $(4.45)_2$ in a form similar to (4.49):

$$A_{1_\alpha} = P_1(h) + P_2(h)\alpha_\alpha$$

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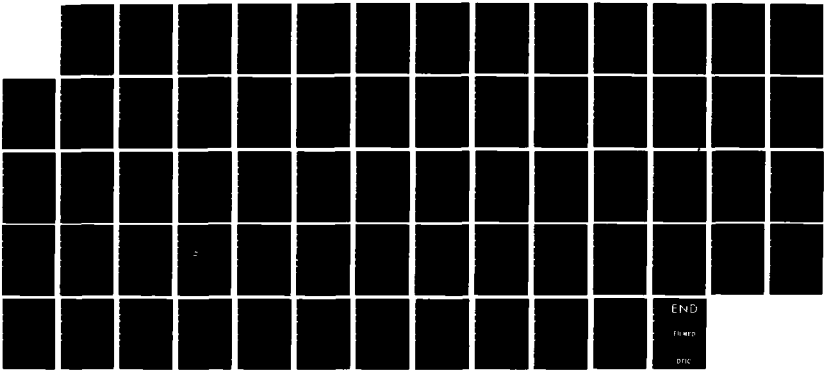
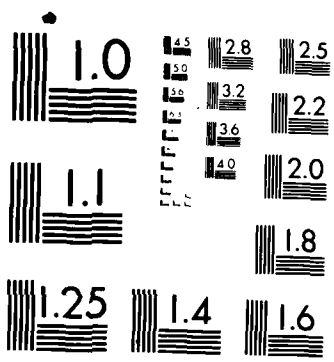




Based on experimental observations, STOUFFER and BODNER [1979] provide the following empirical relations for each term in (4.49) and (4.50);

$$\beta_\alpha = D_0^2 \exp\left[-\frac{n+1}{2n}\left(A_\alpha^2/3J_2\right)^n\right] J_2^{\frac{1}{2}} \tag{4.51}$$

$$A_{1_\alpha} = h_0 + q\int_0^t \dot{h}(\tau)\,d\tau + (1-q)\alpha_\alpha\int_0^t \gamma_\alpha(\tau)\dot{h}(\tau)d\tau \tag{4.52}$$

$$\dot{h} = m(h_1-h_0)\exp[-m\,\lambda/h_0]\dot{\lambda}/h_0 \tag{4.53}$$

with

$$\alpha_\alpha = d_\alpha^P/\,|\,d_\alpha^P \quad , \quad \gamma_\alpha = d_\alpha^P\,/\,\|\underset{\sim}{d}^P\| \tag{4.54}$$

$$\dot{\lambda} = \underset{\sim}{S} : \underset{\sim}{D}^P = \sum_\alpha T_\alpha\, d_\alpha^P \tag{4.55}$$

Here the coefficients D_0, n, h_0, h_1, m and q (the latter specifying the degree of isotropic hardening) are material constants, γ_α represents the cosine of the angle between the loading direction and the principal hardness direction, and $\|\cdot\|$ denotes the Euclidean norn.

4.4.5.3 *Calculation of Yield Surfaces*

An interesting by-product of the plasticity theories discussed here is that one can calculate an equivalent yield surface by setting an arbitrary permanent strain level as an indication of yielding. In fact, this rather arbitrary selection of a measure of yielding is no different than what is done experimentally to determine yield stresses for most metals.

In the present cases we suppose that we are supplied with a specimen of material characterized by a collection of constitutive equations of the Bodner-type [e.g. (4.51) - (4.55)] and that an essentially uniform state of stress is developed during a laboratory test (e.g., a uniaxial tension test of a cylindrical bar of the material).

We shall assume that yielding of the material occurs whenever a .2 percent offset permanent strain is experienced. We shall thereby trace out an effective yield surface for this material by determining the yield stress in various directions and connecting these stress points in the π-plane

A numerical procedure for performing this construction is outlined as follows:

A) Numerical Scheme

An explicit method for integration of rate type-variables is used for convenience. The following algorithm is used:

(1) specify as an increment of stress,

$$\underset{\sim}{T}^n = n \, \underset{\sim}{T}_{max}/N \tag{4.56}$$

where N is total number of increments and $\underset{\sim}{T}_{max}$ is the maximum applied load.

(2) Calculate the increment of the hardness variable h,

$$\Delta h^n = m(h_0 - h_1) \exp[-m \, \lambda^{n-1}/h_0]\Delta\lambda^{n-1}/h_0 \tag{4.57}$$

$$\Delta\lambda^{n-1} = \sum_i T_i^{n-1}\Delta P_i^{n-1} \text{ and } \lambda^{n-1} = \sum_{k=1}^{n-1} \Delta\lambda^k \tag{4.58}$$

(3) Calculate A_α

$$A_\alpha^n = h_0 + q \sum_{k=1}^{n} \Delta h^k + (\gamma q)\ \text{sign}[\Delta\varepsilon_i^P] \sum_{k=1}^{n} \Delta h^n \gamma_\alpha^n \tag{4.59}$$

(4) Calculate the plastic moduli

$$\beta_\alpha^n = D_0^2 \exp[-\frac{n+1}{2n}(A_\alpha^{n\,2}/3J_2^n)]J_2^{n\,\frac{1}{2}} \tag{4.60}$$

(5) Calculate the plastic strain increment

$$\Delta P_i^n = \beta_\alpha^n T_\alpha^n \tag{4.61}$$

Next we proceed to the calculation of the yield stress. During this calculation, the hardness variables A_α are assumed to be constant since up to .2 percent plastic strain there should be no significant change in A_α.

(6) Increase the stress in various directions until .2 percent of an appropriate measure of total strain $\|P\|$ is attained.

$$\Delta P_i^n = D_0^2 \exp[-\frac{n+1}{2n}(A_\alpha^{n\,2}/3J_2^n)]J_2^{n\,\frac{1}{2}} T_\alpha^n \tag{4.62}$$

(7) Project the yield stress into the π-plane as follows:

(a) First, take stress vectors which lie in the σ_1 - σ_2 plane, since we change the direction of stress vector in the σ_1 - σ_2 plane for convenience in step (6). Then calculate the magnitude

of that vector and the angle between the vector and the σ_1 axis (see Fig. 4.3):

$$\sigma = |\underset{\sim}{\sigma}| = \sqrt{\sigma_1^2 + \sigma_2^2}$$

$$\theta = \text{Arc tan } (\sigma_2/\sigma_1)$$

(b) Project $\underset{\sim}{\sigma}$ into the intersection line and the line perpendicular to the intersection line which lies in $\sigma_1 - \sigma_2$ plane,

$$\sigma_s = |\underset{\sim}{\sigma}| \sin (\tfrac{\pi}{4} + \theta)$$

$$\sigma_c = |\underset{\sim}{\sigma}| \cos (\tfrac{\pi}{4} + \theta)$$

(c) Project σ_s into π-plane and compute the magnitude of the projection σ_π of σ_s ,

$$\sigma_{\pi_1} = \sigma_s \cos \psi, \quad \sigma_{\pi_2} = \sigma_c$$

and

$$\sigma_\pi = \sqrt{\sigma_{\pi_1}^2 + \sigma_{\pi_2}^2}$$

where $\phi = \cos^{-1} (1/\sqrt{3})$

(d) Compute the angle between the line perpendicular to the intersection line in the π-plane,

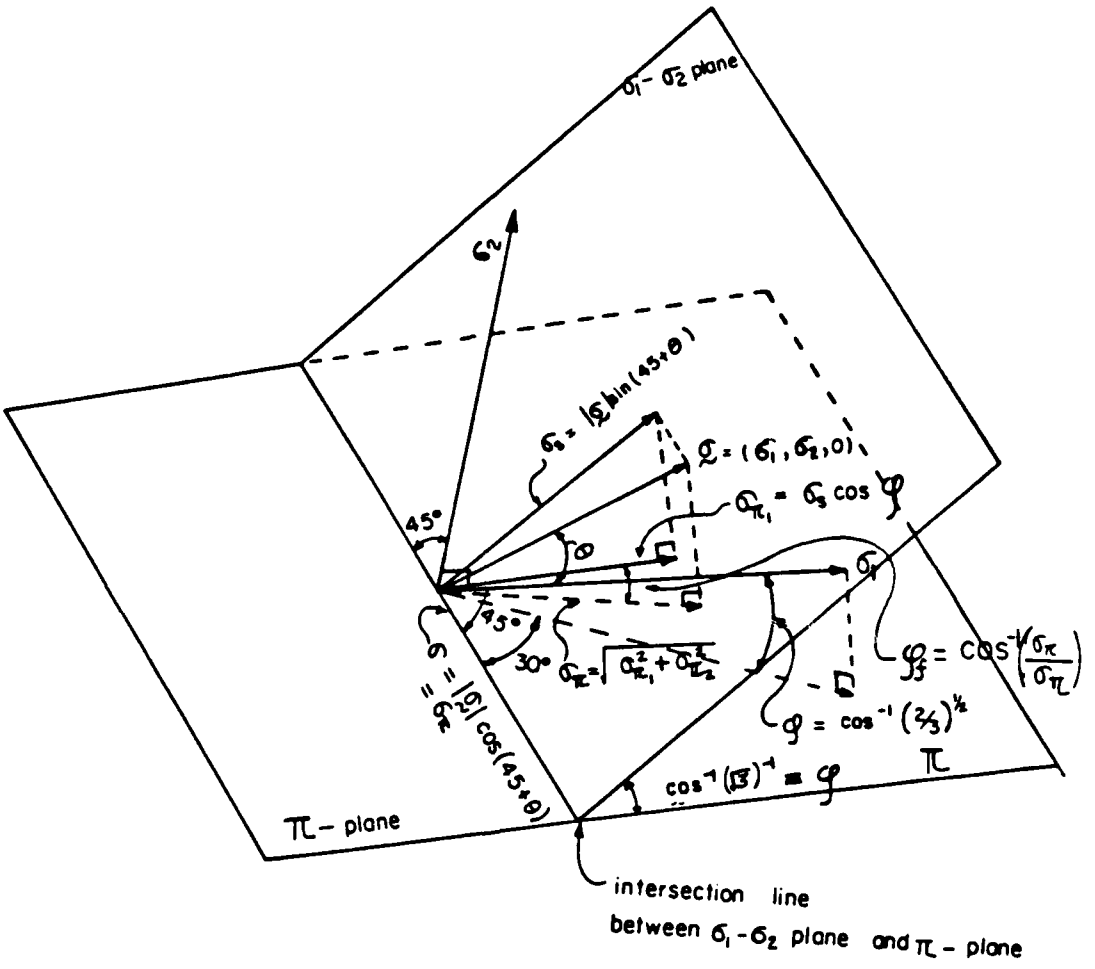


Figure 4.3 Projection of the stress vector into the π-plane.

$$\phi_f = \cos^{-1}(\phi_{\pi_1}/\sigma_\pi)$$

Throughout the steps (a) - (d), the yield stress can be placed in the π-plane.

B) Numerical Experiments

First, we consider a specimen of titanium RMI-50A. The material constants of this specimen listed in BODNER and PARTOM [1975] are:

$m = 100$

$n = 1$

$D_0 = 10^4 \text{ sec}^{-1}$

$h_0 = 1150 \text{ N/mm}^2$ (= 167.9 Kpsi)

$h_1 = 1450 \text{ N/mm}^2$ (= 211.7 Kpsi)

The elastic constants are taken to be

$K = 123{,}000 \text{ N/mm}^2$

$G = 44{,}000 \text{ N/mm}^2$

To check our algorithm, the stress-strain curve produced in a uniaxial test with the maximum loading $T_{max} = 400 \text{ N/mm}^2$ was calculated and the results are shown in Fig. 4.4.

Next a yield surface is created after a uniaxial loading history for the case $q = 1$ and $q = 0$ as indicated in Fig. 4.5 and 4.6, respectively. In Fig. 4.5, we recognize that the Bodner et al theory is equivalent to the Levy-Mises yield criterion in the

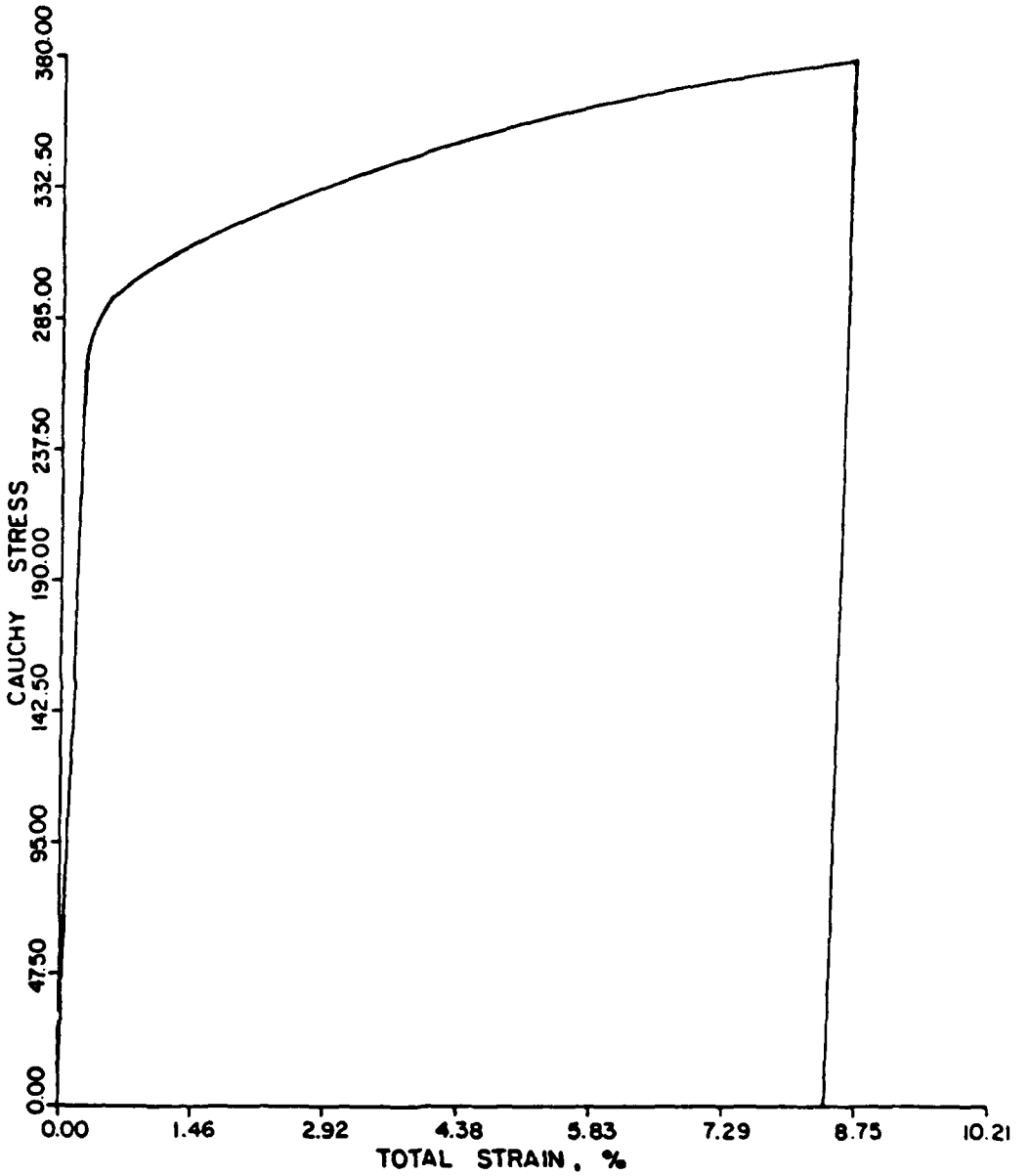


Figure 4.4 Computed engineering strain-stress curve in uniaxial test.

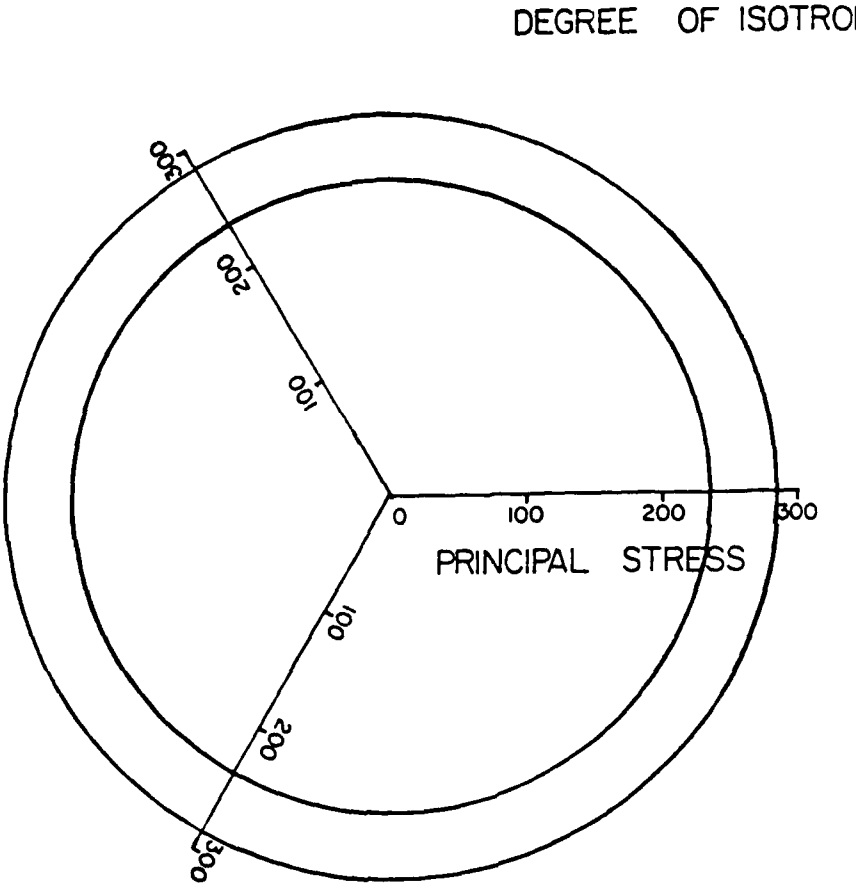


Figure 4.5 Computed yield surfaces in the π-plane for isotropic hardening; the inner circle is the initial yield surface.

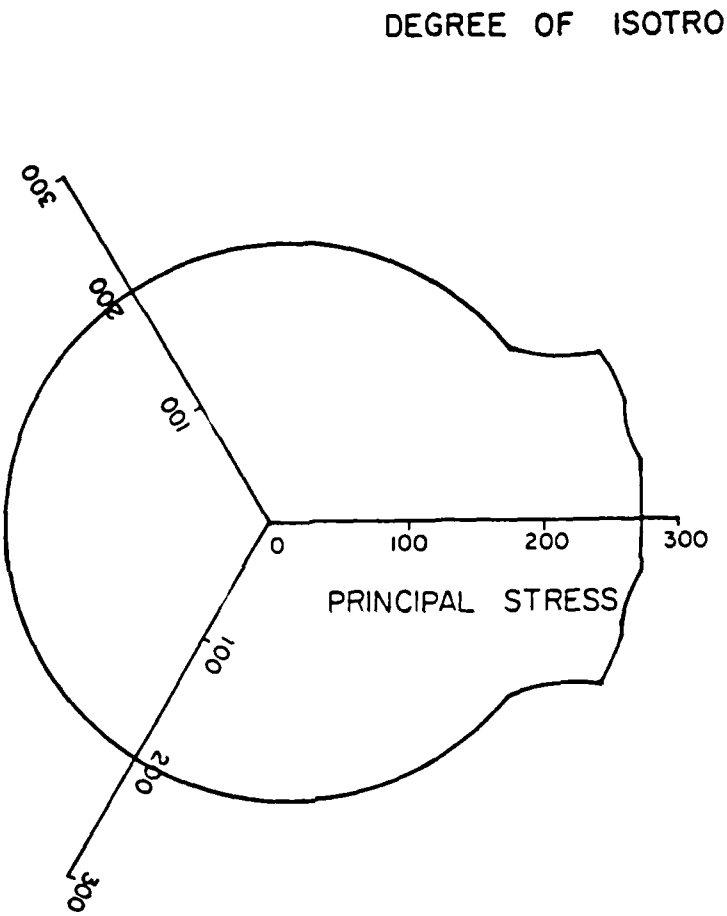


Figure 4.6 Subsequent yield surface after uniaxial load with non-isotropic hardening.

case of q = 1. Quite different results are attained in the hardening case illustrated in Fig. 4.6; not only do corners appear, but a marked lack of convexity is seen to appear as the yield surface assumes a "key-hole" shape.

4.4.6 *Case VI: Modified Bodner and Partom Theory*

In the previous section, we analyzed the constitutive equations by Bodner and his coworkers. In their theory, the variable λ conjugate to the hardness variable h needed to be defined but in our theory of materials of type N this variable can be derived from the flow potential ψ. Here we will furnish the theory which follows the theory of materials of type N, that implies thermo-mechanical validity, and we will use these constitutive equations for the finite element computations in the next chapter.

4.4.6.1 *Equations*

We first introduce a pair of potentials as in the previous cases:

$$\phi = \frac{1}{2}[\frac{\lambda}{\rho_o}\,(\mathrm{tr}\underset{\sim}{E})^2 + \frac{2\mu}{\rho_o}\,(\mathrm{tr}\underset{\sim}{E})^2] - h_1\lambda - \frac{1}{m}\,(h_1-h_o)\,\exp\,(-m\lambda) \tag{4.63}$$

and

$$\psi = Do \sum_{i=o}^{\infty} (-1)^{i+1}\,\frac{B^i}{i!(2ni-\frac{1}{2})}\,\frac{h^{2ni-1}}{J_2^{ni-\frac{1}{2}}} \tag{4.64}$$

Where $B = \frac{1}{3n}(\frac{n+1}{n})$ and λ, μ are Lame's constants and D_o, h_o, h_1 and n are material constants.

Applying (3.48) and results (3.61) and (3.70) to the potentials, we obtain the following elastoplastic constitutive relations.

$$\underset{\sim}{S} = \rho \frac{\partial \phi}{\partial \underset{\sim}{E}} = \frac{\rho\lambda}{\rho_o} (tr\underset{\sim}{E})\underset{\sim}{1} + \frac{\rho}{\rho_o} 2\mu\underset{\sim}{E} \tag{4.65}$$

$$h = - \frac{\partial \phi}{\partial \lambda} = h_1 + (h_o - h_1) \exp(-m\lambda) \tag{4.66}$$

$$\underset{\sim}{P} = \frac{\partial \psi}{\partial \underset{\sim}{S}} = D_o \frac{1}{h\sqrt{J_2}} \exp\left(- \frac{Bh^{2n}}{J_2^{\,n}}\right) \underset{\sim}{S}' \tag{4.67}$$

$$\lambda = - \frac{\partial \psi}{\partial h} = \frac{1}{h} \underset{\sim}{S}' : \underset{\sim}{P} \tag{4.68}$$

where $\underset{\sim}{S}' = \underset{\sim}{S} - \frac{1}{3} tr(\underset{\sim}{S})$

From the thermodynamic restriction, we obtained the inequality (3.71). We need to check whether the given constitutive relations satisfy this inequality. Otherwise, we have to impose this inequality to get a correct set of solutions.

$$\underset{\sim}{S} : \dot{\underset{\sim}{P}} = \underset{\sim}{S} : \left[D_o \frac{1}{h\sqrt{J_2}} \exp\left(- \frac{Bh^{2n}}{J_2^{\,n}}\right)\right] \underset{\sim}{S}'$$

$$= [\text{positive value}]\ (2J_2) \geq 0$$

$$- \frac{\partial \phi}{\partial \underset{\sim}{\alpha}} : \dot{\underset{\sim}{\alpha}} = \frac{\partial \phi}{\partial \lambda} : \dot{\lambda}$$

$$= h\dot{\lambda} = \underset{\sim}{S}' : \dot{\underset{\sim}{P}} \geq 0$$

Thus, $- (1/\theta)\ \underset{\sim}{q} \cdot \nabla\theta \geq 0$, as generally required.

4.4.6.2 *Determination of Coefficients from Experimental Results*

We now have a complete set of constitutive equations for elastoplasticity. A natural question that arises here is how to determine the material constants experimentally.

To determine the elastic constants λ and μ (or Young's modulus E and shear modulus G), we need two experiments, i.e., tension test and torsion test in the infinitesimal elastic strain range.

For the plastic parts, there are 5 constants to be determined, i.e., D_o, h_o, h_1, m and n. To determine these constants we need at least two uniaxial tension (or compression) tests with different strain rates since the constitutive equations have a capability to model rate sensitivity. Next, let us examine the role of individual coefficients.

A) D_o acts as a scaling factor to the plastic strain rate. It also effects the yield stress in a classical sense.

B) h_o and h_1 are minimum and maximum values of hardness variable. h_o will mainly determine the yield stress and h_i will limit the maximum achievable stress (actually J_2).

C) m adjusts the growth rate of the hardness variable h with respect to λ.

D) n determines the hardening tendency of the material. We may obtain a elastic-perfect-plasticity as $n \to \infty$. Usually $n = 1$ is chosen for the hardening material since the detailed approximation can also be done through the previous coefficients.

Next we explain how to determine those material constants from experimental data. The stress strain curves in Fig. 4.7 were taken from the experiments done by Instron 1125 test machine with pure lead (more than 99%) specimens which have 25 mm gauge length with cross-sections size 3.8 mm x 6.5 mm.

An Instron extensometer was used for strain (in fact, the total displacement of gauge length) measurement and the load was recorded as a function of the strain. The crosshead velocities applied were 0.5, 5 and 200 mm/min. which, for an effective overall gauge length of 25 mm, correspond, respectively, to the constant engineering strain rate of 3.3×10^{-4}, 3.3×10^{-3}, and 1.33×10^{-1}, respectively. The experimental curves at the lowest and the fastest rates are chosen to determine the material constants.

First we choose $n = 1$ as a common choice for hardening material. Let a uniaxial stress in the experiment be σ, i.e.

$$\sigma_x = \sigma \quad , \sigma_y = 0 \quad , \quad \sigma_z = 0 \tag{4.69}$$

and

$$\sigma_x' = 2\sigma/3 \quad , \quad \sigma_y' = -\frac{\sigma}{3} \quad , \quad \sigma_z' = -\frac{\sigma}{3}$$

$$J_2 = \frac{1}{2}\left[\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right] \sigma^2 = \frac{\sigma^2}{3} \tag{4.70}$$

Thus the constitutive equations (4.65) - (4.68) in this state of stress become simplified as follows:

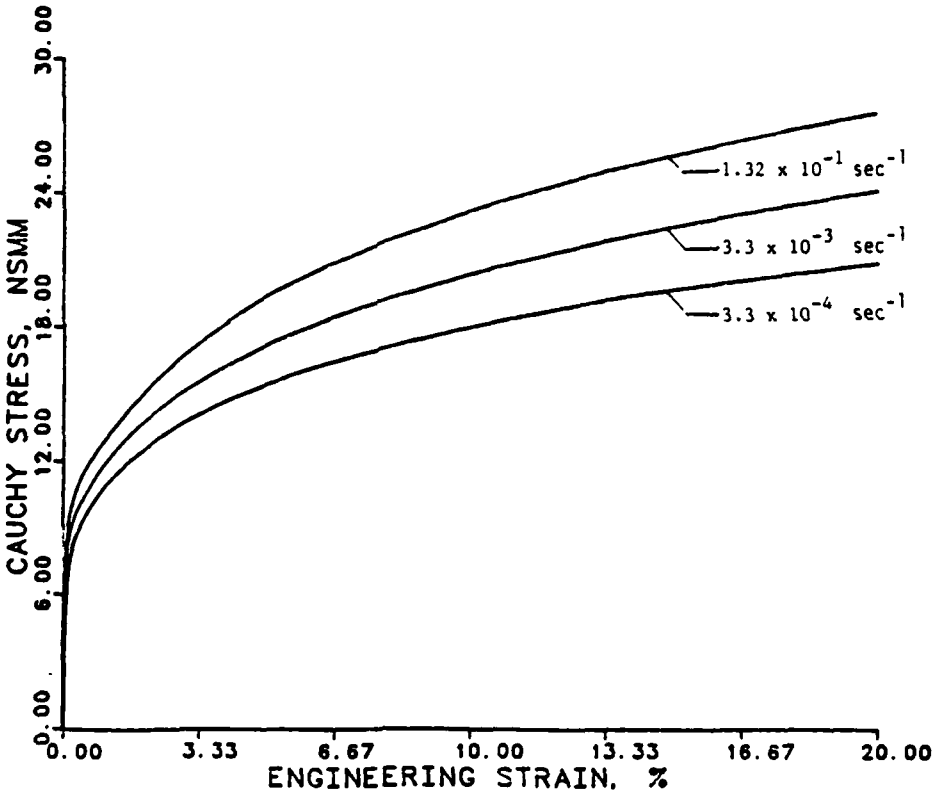


Figure 4.7 Stress-strain curves of pure lead from laboratory tension tests for various strain rate.

$$S = \sigma_x = \frac{\rho}{\rho_o} \lambda(E_1 + E_2 + E_3) + \frac{\rho}{\rho_o} 2\mu E,$$

or

$$E_1 = \frac{\rho_o C}{\rho} \sigma \tag{4.71}$$

$$h = h_i + (h_o - h_i) \exp(-m\lambda) \tag{4.72}$$

$$\dot{P}_1 = \frac{2\sqrt{3}D_o}{3h} \exp\left(-\frac{2h^2}{\sigma^2}\right) \tag{4.73}$$

$$\dot{\lambda} = \frac{1}{h} \sigma \dot{P}_1 \tag{4.74}$$

To determine the constants h_o and D_o, we utilize a conventional definition of yield stress, i.e., about a 0.2% offset strain. Since $h = h_o$ when yielding starts to occur and two unknowns, h_o and D_o, need be determined, we take two test results at the highest and the lowest strain rate in Fig. 4.7.

REMARK 4.6 In experiment, the engineering strain is defined by $\varepsilon = (L-L_o)/L_o$, L = current length and L_o = Initial length. But the value of our strain definition in this uniaxial situation (for detail, see section 3.3.2) can be found as:

$$\text{Total strain} = E_1 + P_1 \simeq \varepsilon L_o/L$$

With $\rho \simeq \rho_o$ and C = 17900 N/mm^2, we obtain the following values at onset of yielding.

Total strain rate	Plastic strain rate	Cauchy stress
1.32×10^{-1}	1.27×10^{-1}	10.58 N/mm^2
3.3×10^{-4}	3.2×10^{-4}	8.73 N/mm^2

Then from this table and (4.73), we have

$$D_o = 1029400 \text{ N/mm}^2 \text{ sec}$$

$$h_o = 26.73 \text{ N/mm}^2$$

To determine h_1 and m, we need to calculate h. Taking logarithm of (4.73) gives

$$\ell n\ P_1 = \ell n\ \frac{2\sqrt{3}D}{3} - \ell n\ h - \frac{2h^2}{\sigma^2} \tag{4.75}$$

By using the Newton Raphson scheme, we obtain the variation of h versus strain as in Fig. 4.8.

With the computed value h, we are ready to determine h_i and m. First, take time derivative to (4.72).

$$\dot{h} = -m(h_o - h_1)\ \exp\ (-m\lambda)\dot{\lambda} \tag{4.76}$$

And then, from (4.72) and (4.76) we obtain

$$m = \frac{1}{\lambda}\ \ell n\left(\frac{h_1 - h_o}{h - h_o}\right) \tag{4.77}$$

and

$$h_1 = h + \frac{\dot{h}}{m\dot{\lambda}} \tag{4.78}$$

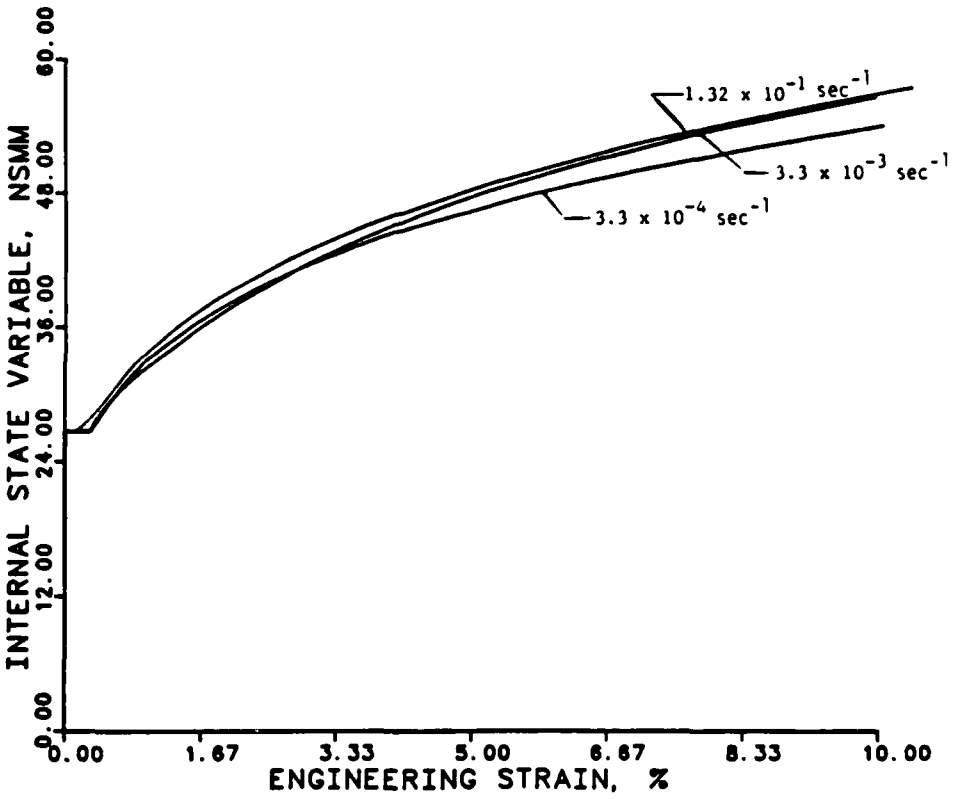


Figure 4.8 Computed variation of the internal state variables from experimental data

We compute m and h_i by an iterative scheme at several loading points and get averages.

$$m = 75$$

$$h_1 = 85 \text{ N/mm}^2$$

Fig. 4.9 shows the experimental and the computed results.

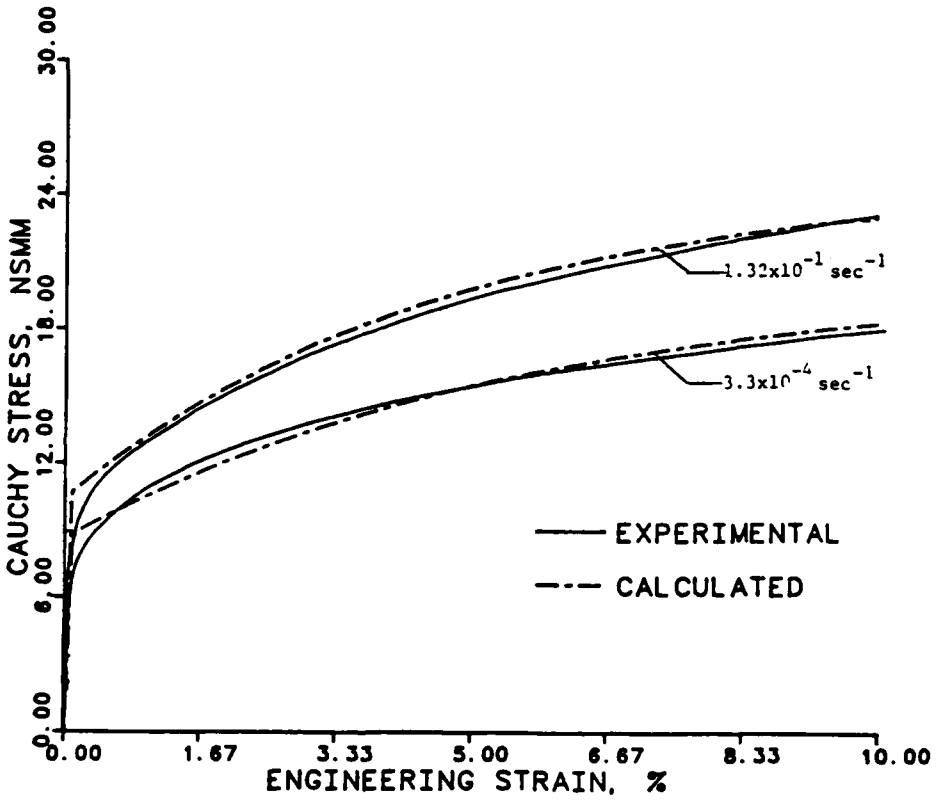


Figure 4.9 Experimental and calculated stress and strain curve

CHAPTER V

FINITE ELEMENT APPROXIMATION

In this chapter, we solve a boundary value problem with the constitutive equations introduced in the Section 4.4.6. We limit ourselves to quasi-static problems involving no body forces and no temperature dependence. That means we do not solve the energy equation directly and that the equation of momentum balance in the reference configuration will be of the form

$$\nabla_{\underset{\sim}{x}} \underset{\sim}{T} = \underset{\sim}{0} \quad \text{in } C_o \tag{5.1}$$

where $\underset{\sim}{T}$ is the first Piola Kirchhoff stress and

$$\nabla_{\underset{\sim}{x}} \overset{\text{def}}{=\!=} \frac{\partial}{\partial \underset{\sim}{x}} .$$

Early accounts of finite element models of elastoplasticity were reported by ODEN and KUBITZA [1967], MARCAL and KING [1967], ARGYRIS [1967], and ZIENKIEWICZ et al [1969], and finite deformations were treated by ODEN [1968, 1970, 1972] and others. A variety of different formulations of the large deformation problem have been explored, and we mention the incremental Lagrangian formulation of HIBBIT, MARCAL and RICE [1970], the updated Lagrangian scheme,of, e.g., MCMEEKING and RICE [1975], and various related schemes proposed by NEEDLEMAN [1972], LARSEN and POPOV

[1974], BATHE, RAMM and WILSON [1975], ARGYRIS et al. [1977, 1978], CESSCOTTO, FREY and FOUNDER [1979] and ATLURI [1980]. Numerous applications of finite element methods to metal forming problems can be found in the literature and we mention as examples those in the papers of LEE, MALLET and YANG [1977], NAGTEGAAL and DeJONG [1980], KEY, KRIEG and BATHE [1979], ARGYRIS and DOLTINIS [1979, 1980], TAYLOR and BECKER [1983], and KIKUCHI and CHENG [1983]. A good survey of current theories and numerical methods for finite-deformation plasticity can be found in the volume edited by NEMET-NASSER, ASARO, and HEGEMIER, [1984] and in the proceedings edited by WILLAM [1984]; see also the recent work of SIMO and MARSDEN [1984], SIMO and ORTIZ [1985], and KIM and ODEN [1985] and the references therein.

5.1 Formulations

Here we adopt the incremental total Lagrangian formulation, where the reference configuration is always the initial configuration. Let $\underset{\sim}{q}$ denote a test function which belongs to a set of V admissible displacement increments. Then a weak form of equation (5.1) can be written

$$\int_{\Omega_0} \underset{\sim\sim}{T} : \nabla \underset{\sim}{q} \, dx = \int_{\partial\Omega_0} \underset{\sim\sim}{T}\underset{\sim}{N} \cdot \underset{\sim}{q} \, dS, \qquad \forall \underset{\sim}{q} \in V \tag{5.2}$$

where $\underset{\sim}{N}$ is the outward normal unit vector in the reference configuration.

We assume that the equilibrium state is achieved at n-1 th incremental step, i.e.,

$$\int_{\Omega_0} \underset{\sim}{T}_{n-1} : \underset{\sim}{\nabla} \underset{\sim}{q} \, d\underset{\sim}{x} = \int_{\partial\Omega_0} \underset{\sim}{T}_{n-1} \underset{\sim}{N} \cdot \underset{\sim}{q} \, dS, \qquad \forall \underset{\sim}{q} \in V \tag{5.3}$$

Then at the nth step,

$$\underset{\sim}{T}_n = \sum_{i=1}^{n} \Delta \underset{\sim}{T}_i = \underset{\sim}{T}_{n-1} + \Delta \underset{\sim}{T}_n$$

$$\underset{\sim}{t}_n = \underset{\sim}{T}_n \underset{\sim}{N} = \underset{\sim}{t}_{n-1} + \Delta \underset{\sim}{t}_n$$

and

$$\int_{\Omega_0} \Delta \underset{\sim}{T}^n : \underset{\sim}{\nabla} \underset{\sim}{q} \, d\underset{\sim}{x} = \int_{\partial\Omega_0} \Delta \underset{\sim}{t}_n \cdot \underset{\sim}{q} \, dS, \qquad \forall \underset{\sim}{q} \in V \tag{5.4}$$

From the definition of the first Piola-Kirchhoff stress and equations (3.68) and (3.72),

$$\begin{aligned} \Delta \underset{\sim}{T} &= \Delta(J \underset{\sim}{R} \underset{\sim}{S} \underset{\sim}{U}^{-1}) \\ &= \Delta(\lambda (\mathrm{tr} \underset{\sim}{E}) \underset{\sim}{F}^{-T} + 2\mu \underset{\sim}{R} \underset{\sim}{E} \underset{\sim}{U}^{-1}) \\ &= \lambda (\mathrm{tr} \Delta \underset{\sim}{E}) \underset{\sim}{F}^{-T} + 2\mu \underset{\sim}{R} \Delta \underset{\sim}{E} \, \underset{\sim}{U}^{-1} \\ &\quad + \lambda (\mathrm{tr} \underset{\sim}{E}) \Delta \underset{\sim}{F}^{-T} + 2\mu \Delta \underset{\sim}{R} \underset{\sim}{E} \underset{\sim}{U}^{-1} + 2\mu \underset{\sim}{R} \underset{\sim}{E} \Delta \underset{\sim}{U}^{-1} \end{aligned} \tag{5.5}$$

By applying the decomposition relation (3.68) and the form (5.5), equation (5.4) becomes

$$\int_{\Omega_0} [\lambda \, tr(\Delta t \, \underset{\sim}{D}_n)\underset{\sim}{F}_n^{-T} + 2\mu \Delta t \, \underset{\sim}{D}_n \underset{\sim}{F}_n^{-T}] : \nabla \underset{\sim}{q} \, dx$$

$$- \int_{\Omega_0} [\lambda (tr\Delta P)\underset{\sim}{F}^{-T} + 2\mu (\underset{\sim}{R} \, \Delta \underset{\sim}{P} \, \underset{\sim}{U}^{-1})]_n : \nabla \underset{\sim}{q} \, dx$$

$$+ \int_{\Omega_0} [\lambda (trE)\Delta \underset{\sim}{F}^{-T} + 2\mu (\Delta \underset{\sim}{R} \, \underset{\sim}{E} \, \underset{\sim}{U}^{-1} + \underset{\sim}{R} \, \underset{\sim}{E} \, \Delta \underset{\sim}{U}^{-1})]_n : \nabla \underset{\sim}{q} \, dx$$

$$= \int_{\partial\Omega_0} \Delta \underset{\sim}{t}_n : \underset{\sim}{q} \, dx \qquad \forall \, \underset{\sim}{q} \in V \tag{5.6}$$

Now we apply the tangent stiffness scheme with the idea of successive approximation during iterations in an incremental step. We arrive at the following linearized form at the i^{th} iteration in the n^{th} increment:

$$\int_{\Omega_0} [\lambda \, tr(\Delta t \, \underset{\sim}{D}_n^i) + 2\mu \, \Delta t \, \underset{\sim}{D}_n^i](\underset{\sim}{F}^{-T})_n^{i-1} : \nabla \underset{\sim}{q} \, dx$$

$$= \int_{\Omega_0} [\lambda (tr\Delta P)\underset{\sim}{F}^{-T} + 2\mu (\underset{\sim}{R} \, \Delta \underset{\sim}{P} \, \underset{\sim}{U}^{-1})]_n^{i-1} : \nabla \underset{\sim}{q} \, dx$$

$$- \int_{\Omega_0} [\lambda (trE)\Delta \underset{\sim}{F}^{-T} + 2\mu (\Delta \underset{\sim}{R} \, \underset{\sim}{E} \, \underset{\sim}{U}^{-1} + \underset{\sim}{R} \, \underset{\sim}{E} \, \Delta \underset{\sim}{U}^{-1})]_n^{i-1} : \nabla \underset{\sim}{q} \, dx$$

$$+ \int_{\partial\Omega_0} \Delta \underset{\sim}{t}_n^i : \underset{\sim}{q} \, ds - \int_{\Omega_0} \Delta \underset{\sim}{T}_n^{i-1} : \nabla \underset{\sim}{q} \, dx,$$

$$\forall \, \underset{\sim}{q} \in V \tag{5.7}$$

Here the last integrals on the right hand side of (5.7) is the left-hand side of (5.6) at the i^{th} iteration and the first two integrals

5.2 Finite Element Approximation

First we interpolate the functions $\Delta \underset{\sim}{u}^i_n$ and $\underset{\sim}{q}$ as follows:

$$\left.\begin{aligned} \Delta \underset{\sim}{u}^i_n &= \underset{\sim}{u}_\alpha \phi_\alpha(\underset{\sim}{x}), \quad \underset{\sim}{u}_\alpha = \begin{Bmatrix} u^1_\alpha \\ u^2_\alpha \end{Bmatrix} \\ \underset{\sim}{q} &= \underset{\sim}{q}_\alpha \phi_\alpha(\underset{\sim}{x}), \quad \underset{\sim}{q}_\alpha = \begin{Bmatrix} q^1_\alpha \\ q^2_\alpha \end{Bmatrix} \end{aligned}\right\} \tag{5.9}$$

where $\phi_\alpha(\underset{\sim}{x})$ is a shape function, we have used the summation convention, and $\underset{\sim}{u}_n = \underset{\sim}{u}_{n-1} + \sum_i^M \Delta \underset{\sim}{u}^i_n$, M being the iteration index. Then

$$\Delta t \, \underset{\sim}{D}^i_n = \frac{1}{2}\left[\frac{\partial \Delta \underset{\sim}{u}^i_n}{\partial \underset{\sim}{x}} (\underset{\sim}{F}^{-1})^{i-1}_n + (\underset{\sim}{F}^{-T})^{i-1}_n \left(\frac{\partial \Delta \underset{\sim}{u}^i_n}{\partial \underset{\sim}{x}}\right)^T\right]$$

$$= \begin{bmatrix} (F1\phi_{\beta,1}+F2\phi_{\beta,2})u^1_\beta & \frac{1}{2}[(F3\phi_{\beta,1}+F4\phi_{\beta,2})u^1_\beta+(F1\phi_{\beta,1}+F2\phi_{\beta,2})u^2_\beta \\ \frac{1}{2}[(F1\phi_{\beta,1}+F2\phi_{\beta,2})u^2_\beta+(F3\phi_{\beta,2}+F4\phi_{\beta,1})u^1_\beta & (F3\phi_{\beta,1}+F4\phi_{\beta,2})u^2_\beta \end{bmatrix} \tag{5.10}$$

of (5.7) are obtained by taking ΔP, ΔR and ΔU^{-1} as functions of U and transferring these to the right-hand side, i.e.,

$$\underset{\sim}{\rho}_n^{i-1} = \frac{\partial \Delta P}{\partial \Delta U}\Big|_n^{i-1} \Delta U_n^i, \quad \Delta R_n^{i-1} = \frac{\partial \Delta R}{\partial \Delta U}\Big|_n^{i-1} \Delta U_n^i, \text{ etc.}$$

Note that ΔP_n^{i-1}, etc. go to zero when the iteration converges since P_n, R_n, etc. are defined in the following fashion:

$$P = P_{n-1} + \sum_i^M \Delta P_n^i \text{, etc....}$$

Adding (5.3) to (5.7) gives

$$\int_{\Omega_0} [\lambda \mathrm{tr}(\Delta t\, D_n^i) + 2\mu\, \Delta t\, D_n^i](F^{-T})_n^{i-1} : \nabla q \, dx$$

$$= \int_{\Omega_0} [\lambda(\mathrm{tr}\Delta P)F^{-T} + 2\mu(R\, \Delta P\, U^{-1})]_n^{i-1} : \nabla q \, dx$$

$$- \int_{\Omega_0} [\lambda(\mathrm{tr}E)\Delta F^{-T} + 2\mu(\Delta R\, E\, U^{-1} + R\, E\, \Delta U^{-1})]_n^{i-1} : \nabla q \, dx$$

$$+ \int_{\partial\Omega_0} t_n^i \cdot q \, ds - \int_{\Omega_0} T^{i-1} : \nabla q \, dx$$

$\forall q \in V$

$$[\underset{\sim}{F}^{-T}]_n^{i-1} = \begin{bmatrix} F1 & F2 \\ F3 & F4 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{\nabla}\phi_\beta = \begin{Bmatrix} \phi_{\beta,1} \\ \phi_{\beta,2} \end{Bmatrix} \tag{5.11}$$

$$F1\phi_{\beta,1} + F2\phi_{\beta,2} = \beta A$$

$$F3\phi_{\beta,1} + F4\phi_{\beta,2} = \beta B$$

$$\Delta t \underset{\sim}{D}_n^i = \begin{bmatrix} \beta A\, u_\beta^1 & \frac{1}{2}\{\beta B\, u_\beta^1 + \beta A\, u_\beta^2\} \\ \frac{1}{2}\{\beta A\, u_\beta^2 + \beta B\, u_\beta^1\} & \beta B\, u_\beta^2 \end{bmatrix} \tag{5.12}$$

To reduce bookkeeping, we note that the terms $\Delta\underset{\sim}{F}_n^{i-1}$ and $(\Delta\underset{\sim}{U}^{-1})_n$ should tend to zero when convergence in each iteration procedure is achieved. It thus would allow us to delete the second integral on the right-hand side of (5.8). Then the discretized equations of (5.8) at each node in nth element would assume the form,

$$\{q^1, q^2\} \int_{\Omega_0^N} \left\{ \lambda TRD \begin{Bmatrix} \alpha A \\ \alpha B \end{Bmatrix} + 2\mu \begin{bmatrix} \{F1\phi_{\alpha,1}\beta A + \frac{1}{2}(F3\phi_{\alpha,1}\beta B + F1\phi_{\alpha,2}\beta B)\}u_\beta^1 \\ \{F2\phi_{\alpha,1}\beta A + \frac{1}{2}(F4\phi_{\alpha,1}\beta B + F2\phi_{\alpha,2}\beta B)\}u_\beta^1 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} \{\frac{1}{2}(F3\phi_{\alpha,1}\beta A + F1\phi_{\alpha,2}\beta A) + F3\phi_{\alpha,2}\beta B\}u_\beta^2 \\ \{\frac{1}{2}(F4\phi_{\alpha,1}\beta A + F2\phi_{\alpha,2}\beta A) + F4\phi_{\alpha,2}\beta B\}u_\beta^2 \end{bmatrix} \right\} dx$$

$$= \{q^1, q^2\} \left\{ \int_{\Omega_0^N} \left(\lambda(P1 + P2) \begin{bmatrix} \alpha A \\ \alpha B \end{bmatrix} + \begin{bmatrix} C11\phi_{\alpha,1} + C21\phi_{\alpha,2} \\ C12\phi_{\alpha,1} + C22\phi_{\alpha,2} \end{bmatrix} \right) dx \right.$$

$$\left. + \int_{\partial\Omega_\sigma} \begin{Bmatrix} (t_1 + \Delta t_1) \\ (t_2 + \Delta t_2) \end{Bmatrix} dS - \int_{\Omega_0^N} \begin{bmatrix} RSU1\phi_{\alpha,1} + RSU3\phi_{\alpha,2} \\ RSU2\phi_{\alpha,1} + RSU4\phi_{\alpha,2} \end{bmatrix} dx \right\},$$

$$\forall \begin{Bmatrix} q_\alpha^1 \\ q_\alpha^2 \end{Bmatrix} \qquad (5.13)$$

where Ω_0^N is an nth element and $\partial\Omega_\sigma$ is the side of the element on which the traction is prescribed. Summations on α and β are implied. Also,

$$[R]_n^{i-1} = \begin{bmatrix} R1 & R3 \\ -R3 & R2 \end{bmatrix}, [\Delta P]_n^{i-1} = \begin{bmatrix} P1 & P3 \\ P3 & P2 \end{bmatrix}, \quad [U]_n^{i-1} = \begin{bmatrix} U1 & U3 \\ U3 & U2 \end{bmatrix}$$

$$[C] = \frac{2\mu}{\text{Det } U} \begin{bmatrix} RP1 \cdot U2 - RP2 \cdot U3 - RP1 \cdot U3 + RP2 \cdot U1 \\ \\ REP \cdot U2 - RP4 \cdot U3 - RP3 \cdot U3 + RP4 \cdot U1 \end{bmatrix}, \quad [RP] = [R]\,[\Delta P]$$

$$[S]_n^{i-1} = \begin{bmatrix} S1 & S3 \\ S3 & S2 \end{bmatrix}, \quad [RSU] = \begin{bmatrix} RSU1 & RSU2 \\ RSU3 & RSU4 \end{bmatrix} = [R]\,[S][U^{-1}]$$

and

$$\alpha A \equiv F1\phi_{\alpha,1} + F3\phi_{\alpha,2}$$

$$\alpha B \equiv F2\phi_{\alpha,1} + F4\phi_{\alpha,2}$$

Again, by setting

$$\alpha C \equiv F3\phi_{\alpha,1} + F1\phi_{\alpha,2}$$

$$\alpha D \equiv F4\phi_{\alpha,1} + F2\phi_{\alpha,2}$$

equation (5.13) can be reduced to the following set of simultaneous equations for each element:

$$\sum_{\beta_1,\beta_2,\ldots\beta_{NN}} {}^{\alpha\beta}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u^1_\beta \\ u^2_\beta \end{Bmatrix} \quad \begin{Bmatrix} F^1_\alpha \\ F^2_\alpha \end{Bmatrix} \tag{5.14}$$

for every $\alpha = \alpha_1, \alpha_2, \ldots \alpha_{NN}$.

Here the α_i's and β_i's are global node numbers in an element and NN is the number of nodes in each element.

In global form,

$$\sum_{ELEM=1}^{N} \sum_{\beta_1,\beta_2\ldots\beta_{NN}} {}^{\alpha\beta}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u^1_\beta \\ u^2_\beta \end{Bmatrix} \quad \begin{Bmatrix} F^1_\alpha \\ F^2_\alpha \end{Bmatrix} \tag{5.15}$$

for every α

where N is the number of elements, ELEM is an element number and

$$\alpha\beta K_{11} = \int_{\Omega_0^N} \{\lambda \cdot \alpha A \cdot \beta A + 2\mu \cdot F1 \cdot \phi_{\alpha,1} \beta A + \mu \cdot \alpha C \cdot \beta B\} \quad dx$$

$$\alpha\beta K_{12} = \int_{\Omega_0^N} \{\lambda \cdot \alpha A \cdot \beta B + 2\mu \cdot F3 \cdot \phi_{\alpha,2} \beta B + \mu \cdot \alpha C \cdot \beta A\} \quad dx$$

$$\alpha\beta K_{21} = \int_{\Omega_0^N} \{\lambda \cdot \alpha B \cdot \beta A + 2\mu \cdot F2 \cdot \phi_{\alpha,1} \beta A + \mu \cdot \alpha D \cdot \beta B\} \quad dx$$

$$\alpha\beta K_{22} = \int_{\Omega_0^N} \{\lambda \cdot \alpha B \cdot \beta B + 2\mu \cdot F4 \cdot \phi_{\alpha,2} \beta B + \mu \cdot \alpha D \cdot \beta A\} \quad dx$$

$$F_\alpha^1 = \int_{\Omega_0^N} \{(\lambda(P_1 + P_2)F1 + C_{11} - RSU1)\phi_{\alpha,1} + (\lambda(P_1 + P_2)F_3 + C_{21} - RSU3)\phi_{\alpha,2}\} dx + \int_{\partial\Omega_\sigma} t_{1_n}^i \phi_\alpha \quad ds$$

$$F_\alpha^2 = \int_{\Omega_0^N} \{(\lambda(P_1+P_2)F2 + C_{12} - RSU2)\phi_{\alpha,1} + (\lambda(P_1 +P_2)F4 + C_{22} - RSU4)\phi_{\alpha,2}\} \, dx + \int_{\partial\Omega_\sigma} t_{2_n}^i \phi_\alpha \quad ds$$

REMARK:

i) In the first iteration at each incremental step, we impose the incremental essential boundary conditions: but after the first iteration, we must impose the zero values.

ii) It is interesting to note that (5.15) reduces to equation for stiffnesses in linear infinitesimal elasticity upon appropriate specializaton. In the case of infinitesimal deformation,

$$\underset{\sim}{F} = 1,\ \underset{\sim}{R} = \underset{\sim}{1},\ \underset{\sim}{U}^{P} = \underset{\sim}{1}$$

Then

$$\alpha A = \phi_{\alpha,1}\ ,\ \alpha B = \phi_{\alpha,2}\ ,\ \alpha C = \phi_{\alpha,2}\ ,\ \alpha D = \phi_{\alpha,1}$$

$$\beta A = \phi_{\beta,1}\ ,\ \beta B = \phi_{\beta,2}$$

Therefore, as expected,

$$^{\alpha\beta}K_{11} = \int_{\Omega^N}[(\lambda + 2\mu)\phi_{\alpha,1}\phi_{\beta,1} + \mu\phi_{\alpha,2}\phi_{\beta,2}]\ \ \underset{\sim}{dx}$$

$$^{\alpha\beta}K_{12} = \int_{\Omega^N}[\lambda\phi_{\alpha,1}\phi_{\beta,1} + \mu\phi_{\alpha,1}\phi_{\beta,2}]\ \ \underset{\sim}{dx}$$

$$^{\alpha\beta}K_{21} = \int_{\Omega^N}[\lambda\phi_{\alpha,2}\phi_{\beta,1} + \mu\phi_{\alpha,1}\phi_{\beta,2}]\ \ \underset{\sim}{dx}$$

$$^{\alpha\beta}K_{22} = \int_{\Omega^N}[(\lambda + 2\beta)\phi_{\alpha,2}\phi_{\beta,2} + \mu\phi_{\alpha,1}\phi_{\beta,1}]\ \ \underset{\sim}{dx}$$

Note that the assembled matrix then becomes symmetric.

In each iteration, the following constitutive routine has to be solved:

$$h_n^i = h_1 + (h_0 - h_1)\exp(-m\Lambda_n^{i-1})$$

$$\Delta\underset{\sim}{P}_n^i = \Delta t_n \dot{\underset{\sim}{P}}(h_n^i, \underset{\sim}{\sigma}_n^{i-1})$$

$$\Delta\underset{\sim}{E}_n^i = \Delta\underset{\sim}{U}\ \underset{\sim}{U}^{-1}\ \Big|\ \text{sym}\Big]_n^i - \Delta\underset{\sim}{P}_i^n$$

$$\Delta \underset{\sim}{S}_n^i = \frac{1}{J^n} [\lambda (tr \Delta \underset{\sim}{E}_n^i) \underset{\sim}{1} + 2\mu \Delta \underset{\sim}{E}_n^i]$$

$$\Delta \Lambda_n^i = \frac{1}{h_n^i} \underset{\sim}{S}_n^{i'} : \Delta \underset{\sim}{P}_n^i$$

In the actual computation, we subdivide the incremented solution $\Delta \underset{\sim}{U}\, \underset{\sim}{U}^{-1}|_{sym}$ by a prescribed number and proceed by using the previous forward Euler method.

Since the previous constitutive equations assume incompressibility in the plastic deformation, care must be taken in choosing a stable element approximation (see NAGTEGALL, PARKS and RICE [1974] and the detailed stability analysis of ODEN et al [1982, 1984]). Here rectangular elements which consist of four 3-node triangles; so-called, four constant strain triangles (4CST element) issued. A mathematical analysis of this 4CST element can be found in KIKUCHI, ODEN, and SONG [1982] and KIKUCHI [1983].

5.3 Numerical Examples

In this section several example problems are solved to verify the algorithm described above.

The proposed set of constitutive equations (equations (4.65), (4.66), (4.67) and (4.68) contains 7 constants to be determined.

from experimental data. The two elastic constants can be determined by standard procedures,but the plastic constants require at least two uniaxial tests at different strain rates.

Following Bodner and Partom, we consider estimated constants on Titinium for our constitutive equations, which are similar to theirs. The material parameters are as follows:

i) Elastic Constants

λ = 93667 N/mm^2 (1 N/mm^2 = 10^6 Pa)

μ = 44000 N/mm^2

ii) Plastic Constants

n = 1

m = 50

D = 1.35×10^7 sec^{-1}

h_0 = 1150 N/mm^2

h_1 = 1450 N/mm^2

As a first computational example the homogeneous plane strain elongaton of a test block is computed. Results are shown in Fig. 5.1. The computed results reflect up to 20 percent engineering strain and show strong sensitivity to strain rates. Note that slower loading results lower yield stress.

A result of loading, unloading and reloading is shown in Fig. 5.2 with a strain rate of 1.5×10^{-3} per second. The so-called

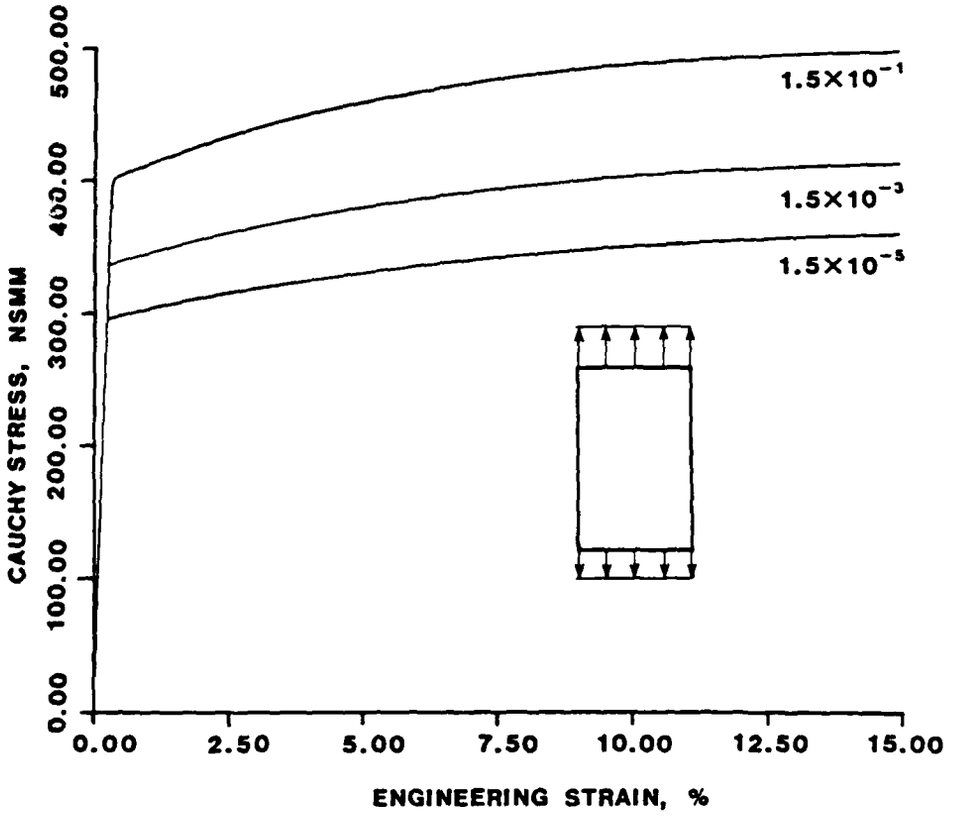


Figure 5.1 Calculated stress-strain curve for titanium for various strain rates.

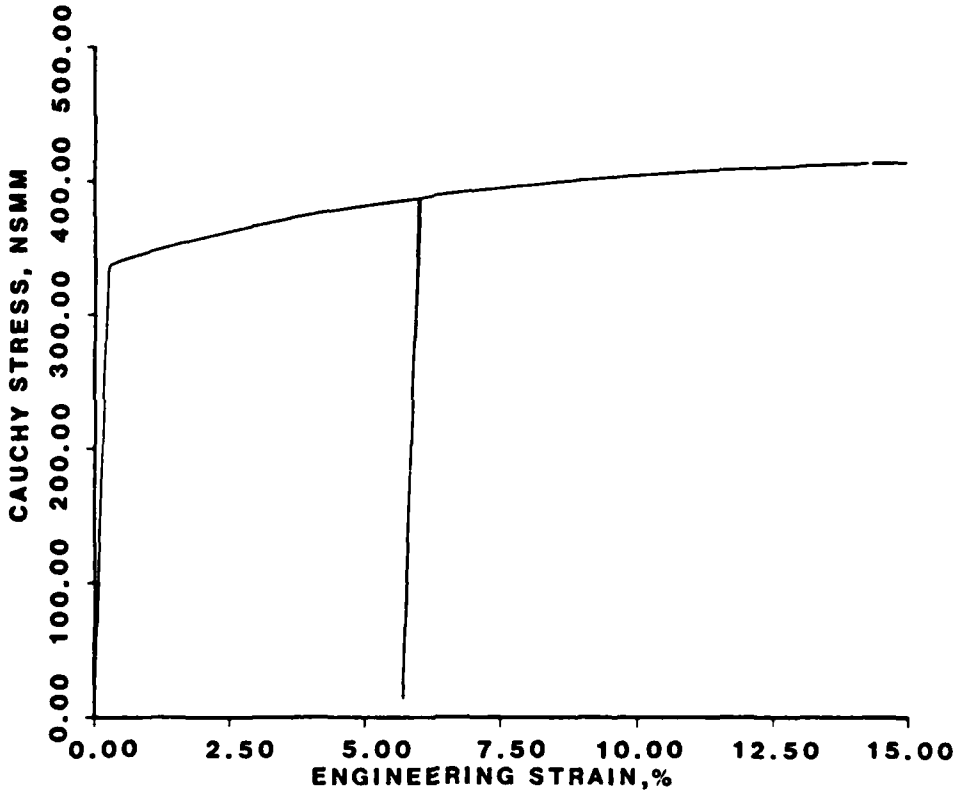


Figure 5.2 Stress-strain curve with loading-unloading-reloading at a strain rate = 1.5×10^{-3}.

ratchet effect at yield point in the reloading process is not apparent because of a large amount of deformation.

In the third example, the variation of the internal state variable with strain is shown in Fig. 5.3. In this case, the internal state variable represents hardness of the material. It is seen to vary with strain in a way which is qualitatively the same as the stress-strain relation.

Before solving a complicated plane strain problem, we next check the algorithm's ability to simulate rotational rigid-body motions. This is done by fixing a corner of the stressed element and prescribing the essential boundary conditions at each incremental step as in TAYLOR and BECKER [1983]. Suppose that a block element, such as that in Fig. 5.4, is subjected to an elasto-plastic deformation according to the following program: We prescribe 0.5 percent engineering strain with a 0.5×10^{-3} per second rate. Next the block is rotated with prescribed incremental rotation angles while maintaining the preloaded deformation. The results are listed in Table 5.1 with increments 9°, 10°, 15°, 30°, and 45° to make 90° rotation. If the computed stress rates are appropriately objective, the stress should not change during these rotations. We observe errors in the stress of only about 0.3 percent and note that there is little error between a large step size (45°) and a small (9°), as expected from the way we defined deformation measures and rotationally-invariant stresses.

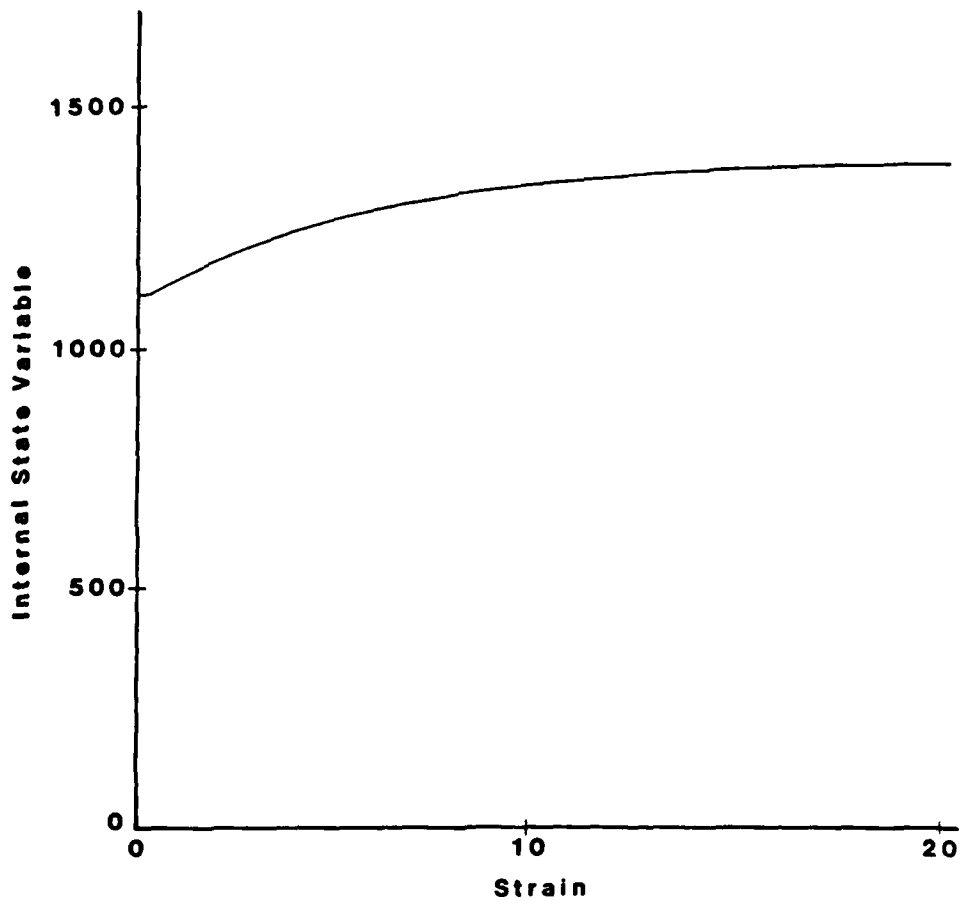
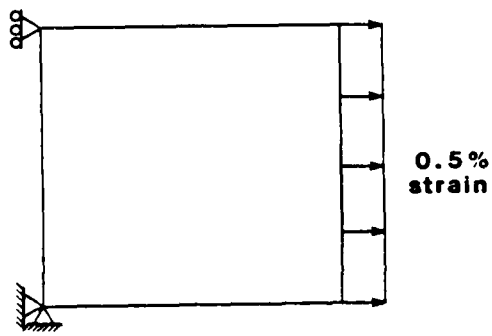


Figure 5.3 Variation of the internal state variable vs. strain.



$S_x = 322.5\ N/mm^2$

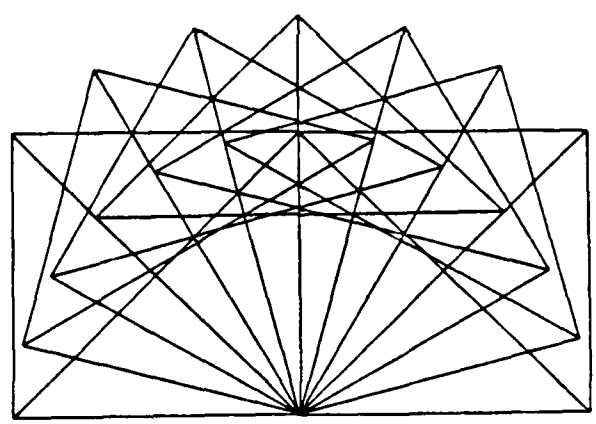
$S_y = 0.\ N/mm^2$

$S_{xy} = 0.\ N/mm^2$

$S_z = 161.3\ N/mm^2$

$J_2 = 80.7\ N/mm^2$

(a)



(b)

Figure 5.4 Calculated rigid body motion with 30° increments,
a) loading before rotation
b) progressive configurations

Table 5.1 Dienes Stress Versus Incremental Rotation Angle

Stress / Increment	S_x	S_y	S_{xy}
9^o	323.4	0.6	0.2
10^o	323.3	0.9	-0.1
15^o	323.6	1.9	0.1
30^o	323.7	0.4	0.1
45^o	323.5	0.1	-0.2

The final example is a head forming problem in plane strain. A 4 by 5 unit rectangular billet, which is confined at the lower boundary is loaded at the upper part without friction. Incremental displacements are prescribed at the five nodes in the top of the billet. Computed, progressive deformed shapes, and J_2 stress contours are shown inFigs. (5.6) and (5.7). Figs. (5.5) and (5.8) show the undeformed and the deformed Lagrangian finite element mesh. The residual J_2 stress contour is shown in the left part of the deformed configuration.

Throughout the finite element computations, the convergence at each incremental step was checked by calculating the maximum relative error of successive incremented displacements. The relative error is computed as the ratio of the correction between iterations to the first solutions (incremental displacements) of the incremental step. A range of tolerances was set as 0.01-1 percent, depending on step size. Generally, the convergence was achieved in two iterations except where severe changes in the deformation from elastic to plastic states are experienced.

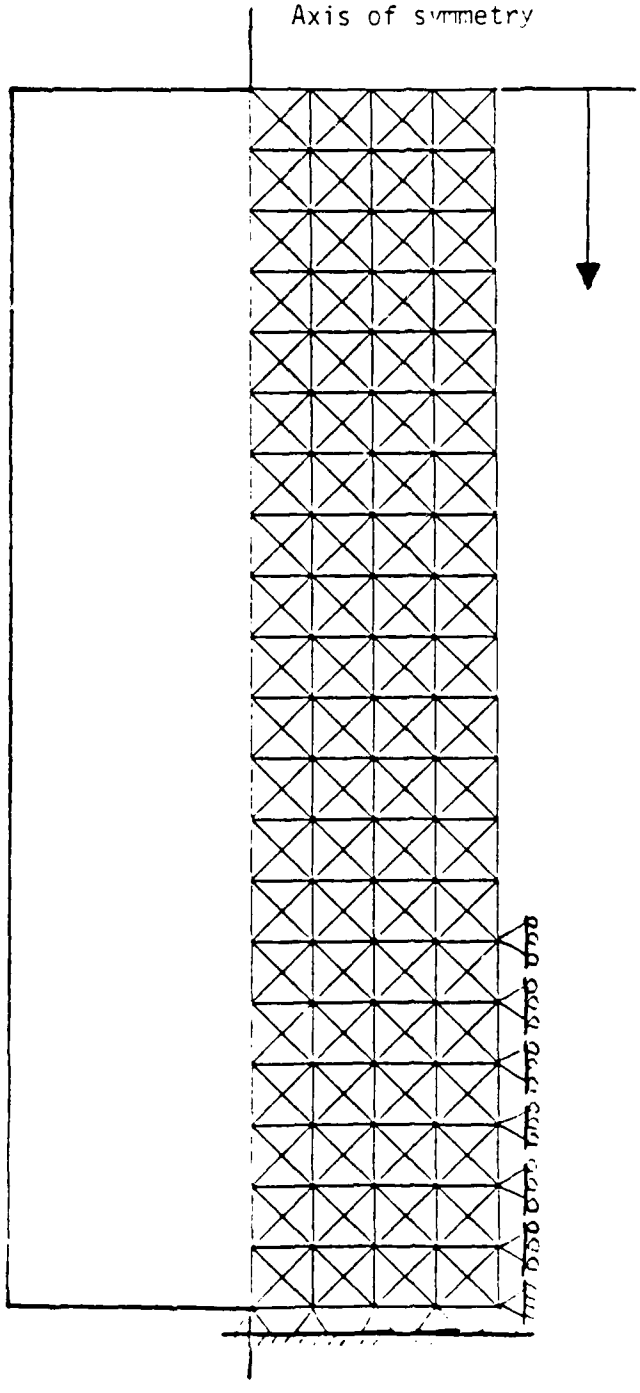
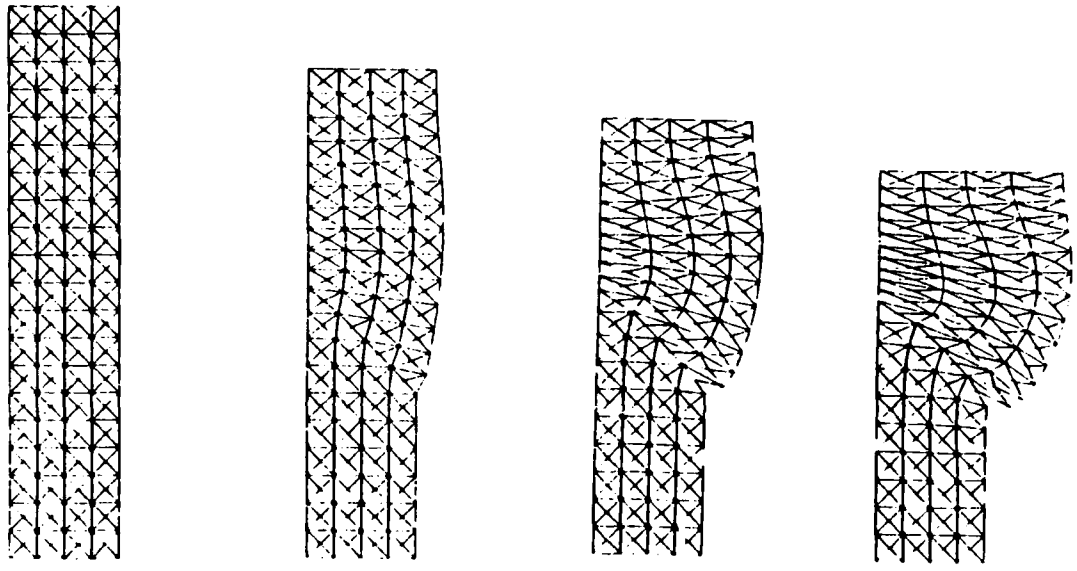


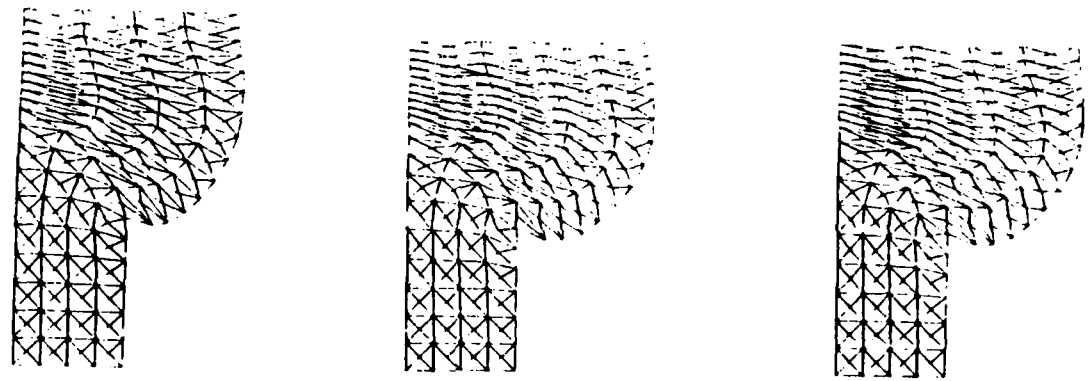
Figure 5.5 Finite element model of a 4 x 5 billet



i) Original Shape ii) 11.2% iii) 20.8% vi) 29.3%

Figure 5.6 Deformed shapes at progressive stages of head forming

(continues to the next page)



v) 35.2° vi) 40° vii) Unloaded Shape

Figure 5.6 continued

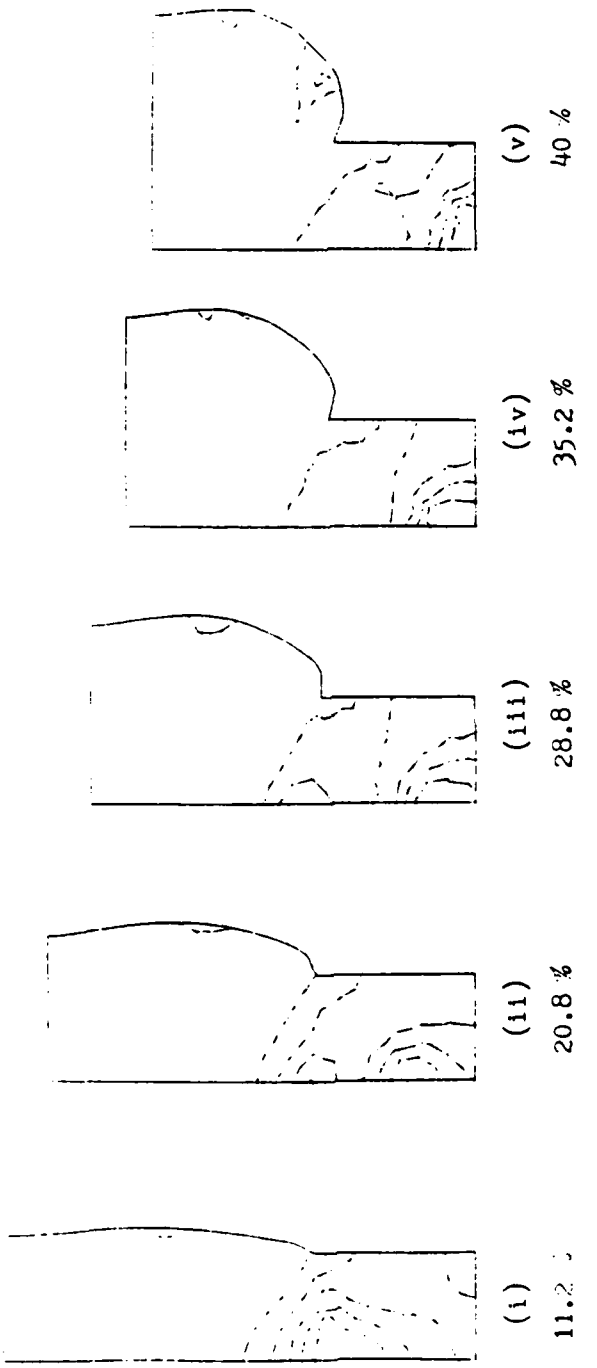


Figure 5.7 J_2 stresses contours at progressive deformation stages

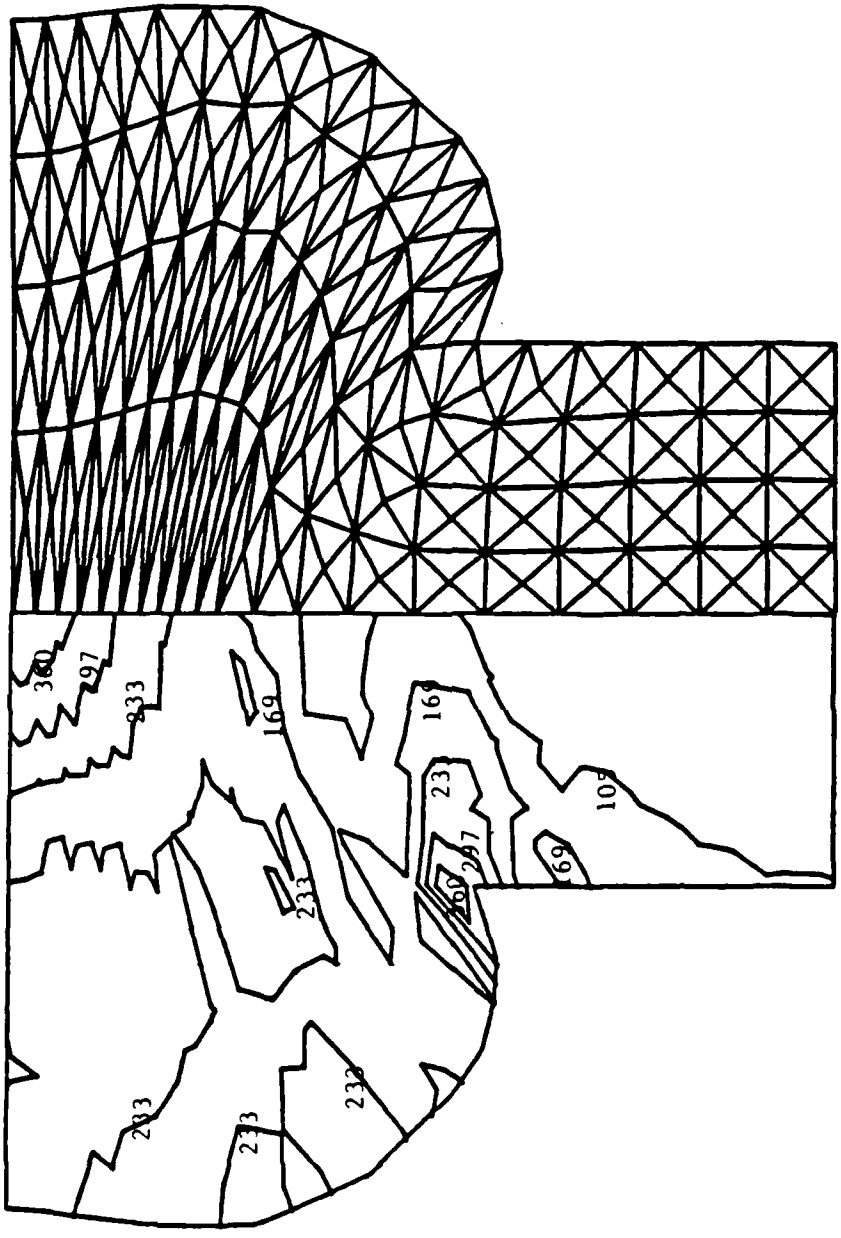


Figure 5.8 Deformed configuration and residual stress (N/mm^2) contour.

CHAPTER VI

CONCLUDING REMARKS

6.1 General

The classical theory of plasticity evolved from simple observations that irrecoverable deformations of many materials may result from a simple load cycle. Characteristic of the classical theory and the developments which arose from it over the last half-century are notions of yield surfaces, hardening laws, and a rather simple kinematics in which the total strain is the sum of so-called elastic and plastic parts. Frequently in modern engineering and manufacturing, situations are encountered in which a more sophisticated constitutive theory and kinematics is needed to describe elastoplastic behavior. This need, and also the natural evolution of plasticity theory as a part of mechanics, has led to some fundamental questions of the mechanical and mathematical foundations of plasticity. For example, how does one develop a theory of elastoplasticity within the framework of modern continuum thermomechanics? How does one obtain a set of constitutive equations which are valid for wide ranges of deformations? What are the deficiencies of existing kinematical measures and is it possible to devise new kinematical measures appropriate

for finite elastoplastic deformation? Can such general theory be put to use to model practical problems and is it possible to devise numerical schemes for implementation of such a broad theory? In this report, an attempt is made to resolve some of these questions. Some success in each area of inquiry has been attained.

6.2 Summary of Results

According to the general aim of this report, the following specific results were obtained.

1. Materials of Type N

A new and general theory of finite elastoplasticity has been developed which has the following features:

a. Involves two potential functionals, one being the free energy functional ϕ and the other the general flow potential ψ.

b. Does not necessarily require the specification of a yield surface.

c. Does not require that ψ be either convex or differentiable.

d. Involves an internal state variable $\underset{\sim}{\alpha}$.

e. Reduces to classical theories as special cases.

f. Is not exclusively a rate-type theory, as the history of microstructural changes is governed by an evolution equation for $\underset{\sim}{\alpha}$.

The material characterized by these features is referred to as a "materials of type N" since the framework generalizes the notion of normality in plasticity.

2. Kinematics of Finite Elastoplasticity

A new kinematical description of notion, valid for finite elastic and plastic deformation and which differs from the theories of Lee and Nemat-Nasser has been derived. Its features are as follows:

a. The effects of rotation are separated from pure stretching.

b. Formalizes the fact that the position vector (or displacement vector) in the intermediate configuration may not be continuous.

c. Results in a correct decomposition of deformation rate into an "elastic" and a "plastic" part.

3. Continuum Thermomechanics and Analysis

a. Analysis via Coleman-Noll thermodynamic methods are used to deduce restrictions on the constitutive equations.

b. It has been shown that the theory results in a fully-determined system of equations governing the thermo-mechanical behavior of materials of type N.

c. Isotropic representations of flow potential and the free energy for the elastoplastic material of materials of type N has been developed and studied.

4. Examples of Materials of Type N

a. Classical plasticity theories are shown to be special simple cases of a materials of type N.

b. An analyses of some newly developed constitutive equations, which do not involve yield functions, is given.

c. Existing constitutive equations for various plasticity theories can be shown to fit within the theory of materials of type N upon appropriate modification.

5. Finite Element Computations

a. Chooses a modified version of the Bodner and Partom equations for finite element approximation.

b. Numerical algorithms and a code are developed for solving a total Lagrangian formulation of the general theory.

c. A test problem of rotation of a prestressed block and head forming problem are solved.

6.3 Future Research

1. Further study is needed to verify that other useful constitutive equations, e.g., by HART [1970, 1976], by MILLER [1976] and anisotropic version of BODNER [1985], can be recast in a form covered by the theory of materials of type N. It is anticipated that such modifications are possible and their construction would be very helpful.

2. To predict the useful lifetime or realibility of performance of engineering material, failure or damage accumulation laws need to be developed. It is expected that it may be possible to incorporate these effects in a theory through the introduction of appropriate internal state variables. One possible example of this may be the work by LEMAITRE [1984].

3. Further development of a modular computer code is needed which can simulate more realistic problems and may include temperature effects and appropriate frictional contact conditions, possibly, in a three-dimensional setting.

APPENDIX

ELEMENTS OF CONVEX ANALYSIS

We shall provide here a brief summary of some of the concepts of convex optimization theory which are prerequisite to the ideas described in the body of the report. For more detailed accounts, the books of EKELAND and TEMAN [1976] or ROCKAFELLAR [1970, 1979] or the recent text of ODEN [1985] can be consulted.

We begin by introducing the following notations:

$\overline{\mathbb{R}}$ = the extended real numbers; if $\mathbb{R}$ is the real number system, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

U,V = topological vector spaces

U*,V* = topological dual spaces of U and V respectively

$\langle \cdot,\cdot \rangle_V$, $\langle \cdot,\cdot \rangle_U$ = duality pairing on V* x V and U* x U, respectively; i.e. if $v^* \in V^*$ and $v \in V$, then $v^*(v) \equiv \langle v^*, v \rangle_V$, etc.

It is worthwhile to recall the definition of the limit-superior (lim sup) and the limit-inferior (lim inf) of sequences of real numbers, extended real-valued functions of sequences, and sequences of sets in a topological vector space V.

lim sup/lim inf.

· For $\{a_n\}$ a sequence of real numbers

$$\left.\begin{aligned} \limsup_{n\to\infty} a_n &= \inf_{N\in\mathbb{N}} \sup_{n\geq N} a_n \\ \liminf_{n\to\infty} a_n &= \sup_{N\in\mathbb{N}} \inf_{n\geq N} a_n \end{aligned}\right\} \qquad (A.1)$$

· For $f: \mathbb{R} \to \overline{\mathbb{R}}$

$$\left.\begin{aligned} \limsup_{x'\to x} f(x') &= \inf_{\delta>0} \sup_{0<|x'-x|<\delta} f(x') \\ \liminf_{x'\to x} f(x') &= \sup_{\delta>0} \inf_{0<|x'-x|<\delta} f(x') \end{aligned}\right\} \qquad (A.2)$$

· For $\{A_n\}$ a sequence of subsets of the underlying set of topological space V,

$$\left.\begin{aligned} \limsup_{n\to\infty} A_n &= \bigcap_{m=1}^{\infty} \left(\bigcup_{n=m}^{\infty} A_n\right) \\ \liminf_{n\to\infty} A_n &= \bigcup_{m=1}^{\infty} \left(\bigcap_{n=m}^{\infty} A_m\right) \end{aligned}\right\} \qquad (A.3)$$

For example, if $\{x_n\}$ is a sequence of real numbers which converges to x and f : $\mathbb{R}$, LIM SUP $f(x_n)$ is the supremum of all cluster points of f at x, as indicated in Fig. A.1 (with an analogous interpretation for lim inf).

The concept can also be applied to multifunctions from one topological vector space to another. Indeed, if $\Gamma : U \to V$ (with $\Gamma(u)$ a subset of V for each vector $u \in U$), then

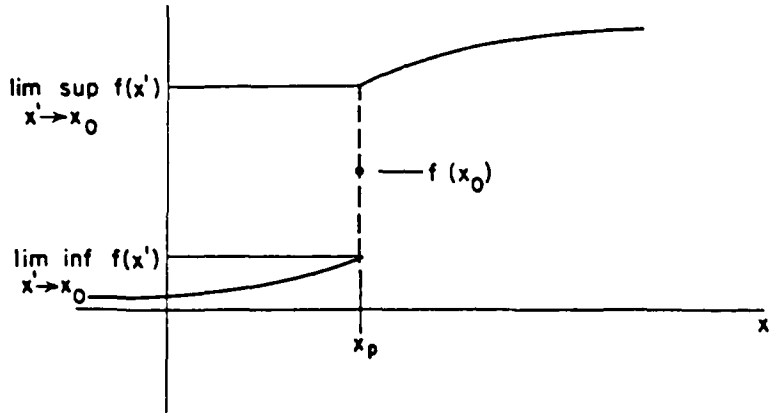


Figure A.1. Limit superior, limit inferior of discontinuous function $f: \mathbb{R} \to \overline{\mathbb{R}}$ at x_0.

$$\limsup_{u' \to u} \Gamma(u') = \bigcap_{A \in N(0)} \bigcap_{B \in N(u)} \left[\bigcup_{u' \in B} (\Gamma(u') + A) \right] \tag{A.4}$$

and

$$\limsup_{u' \to u} \Gamma(u') = \bigcap_{A \in N(0)} \bigcup_{B \in N(u)} \bigcap_{u' \in B} (\Gamma(u') + A) \tag{A.5}$$

where $N(0)$ and $N(u)$ are collections of neighborhoods of 0,u, respectively.

Lim sup inf/Lim inf sup

In addition to the notion of limit superior and limit inferior, it is convenient to introduce the concepts of lim sup inf and lim inf sup introduced by ROCKAFELLAR [1980].

Let F be an extended real-valued function from U x V into $\bar{\mathbb{R}}$, let $u' \to u$ in U and $v' \to v$ in V. Then we define

$$\operatorname*{Lim\,sup\,inf}_{u' \to u, v' \to v} F(u'v') \triangleq \sup_{B \in N(v)} \inf_{A \in N(u)} \sup_{u' \in A} \inf_{v' \in B} F(u',v') \tag{A.6}$$

Likewise,

$$\operatorname*{Lim\,inf\,sup}_{u' \to u, v' \to v} F(u',v') \triangleq \inf_{B \in N(v)} \sup_{A \in N(u)} \inf_{u' \in A} \sup_{v' \in B} F(u',v') \tag{A.7}$$

Similarly, Lim sup sup and Lim inf inf can be defined in an analogous way.

The meaning of these operations can be more easily understood in the case of a real-valued function F defined on $\mathbb{R}^2$, such as the discontinuity at the origin shown in Fig. A.2. To compute

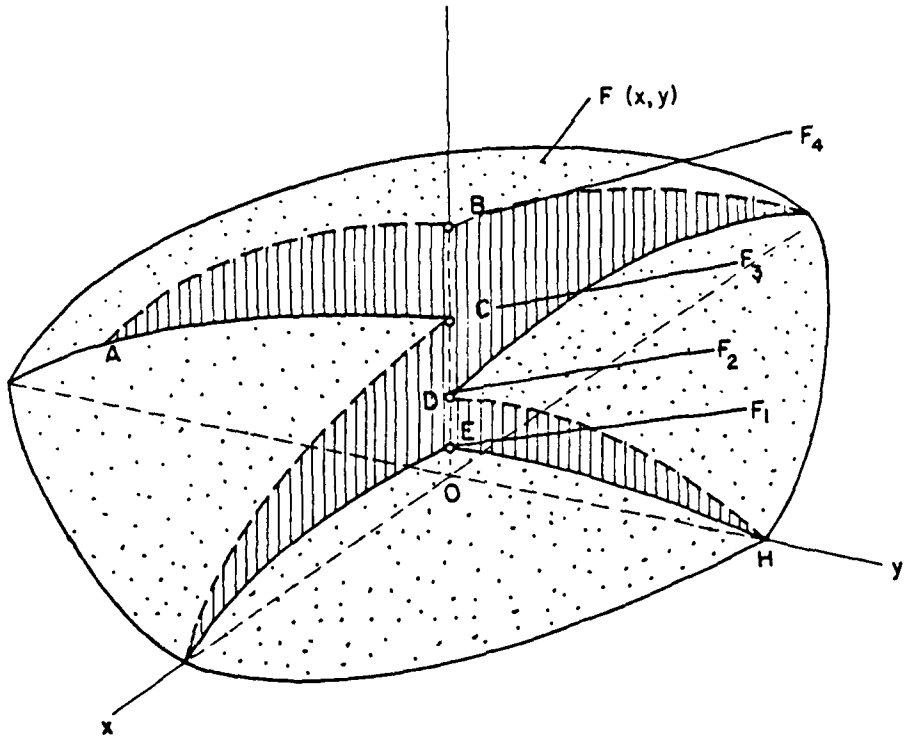


Figure A.2. A function F discontinuous at the origin 0: solid lines at surfaces of discontinuity indicate values assumed by F.

Lim inf inf $F(x',y)$, for example, we compute lim inf $F(x',y)$ for
$x' \to 0$, $y' \to 0$ $x' \to 0$

a fixed y'. This gives the a function of y' which has as its graph

the curve $\overline{AC} \cup \overline{EH}$. The lim inf of this function is the point E,
$y' \to 0$

denoted F_1 in the figure. Similarly lim sup $F(x',y')$ for fixed y',
$x' \to 0$

is the curve, $\overline{HD} \cup \overline{BA}$ and lim inf of this curve is the point D,
$y' \to 0$

denoted F_2 in the figure. In summary for this example,

$$\underset{x' \to 0}{\text{Lim inf}} \; \underset{y' \to 0}{\text{inf}} \; F(x',y') = F_1$$

$$\underset{x' \to 0}{\text{Lim sup}} \; \underset{y' \to 0}{\text{inf}} \; F(x',y') = F_2$$

$$\underset{x' \to 0}{\text{Lim inf}} \; \underset{y' \to 0}{\text{sup}} \; F(x',y') = F_3$$

$$\underset{x' \to 0}{\text{Lim sup}} \; \underset{y' \to 0}{\text{sup}} \; F(x',y') = F_4$$

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